Risk-Averse Decision Making and Control

Marek Petrik
University of New Hampshire

Mohammad Ghavamzadeh
Adobe Research
Outline

Introduction to Risk Averse Modeling

(Average) Value at Risk

Coherent Measures of Risk

Risk Measures in Reinforcement Learning

Time consistency of in reinforcement learning

Summary
# Schedule

<table>
<thead>
<tr>
<th>Time</th>
<th>Session</th>
</tr>
</thead>
<tbody>
<tr>
<td>9:00–9:20</td>
<td>Introduction to risk-averse modeling</td>
</tr>
<tr>
<td>9:20–9:40</td>
<td>Value at Risk and Average Value at Risk</td>
</tr>
<tr>
<td>9:40–9:50</td>
<td><em>Break</em></td>
</tr>
<tr>
<td>9:50–10:30</td>
<td>Coherent Measures of Risk: Properties and methods</td>
</tr>
<tr>
<td>10:30–11:00</td>
<td><em>Coffee break</em></td>
</tr>
<tr>
<td>11:00–12:30</td>
<td>Risk-averse reinforcement learning</td>
</tr>
<tr>
<td>12:30–12:40</td>
<td><em>Break</em></td>
</tr>
<tr>
<td>12:40–12:55</td>
<td>Time consistent measures of risk</td>
</tr>
</tbody>
</table>
Risk Aversion

Risk (Wikipedia):

*Risk* is the potential of gaining or *losing* something of value. . . . *Uncertainty* is a potential, unpredictable, and uncontrollable outcome; *risk* is a consequence of action taken in spite of uncertainty.

Risk aversion (Wikipedia):

. . . *risk aversion* is the behavior of humans, when exposed to uncertainty, to attempt to reduce that uncertainty. . . .

Tutorial: Modern methods for risk-averse decision making
Desire for Risk Aversion

- Empirical evidence:
  1. People buy insurance
  2. Diversifying financial portfolios
  3. Experimental results

- Other reasons:
  - Reduce contingency planning
Where Risk Aversion Matters

- Financial portfolios
- Health-care decisions
- Agriculture
- Public infrastructure
- Self-driving cars?
Introduction to Risk Averse Modeling

When Risks Are Ignored . . .

Seawalls overflow in a tsunami

Housing bubble leads to a financial collapse

The New York Times
Need to Quantify Risk

- Mitigating risk is expensive, how much is it worth?

Expected utility theory:

\[ E[u(X)] = E[\text{utility}(X)] \]

Exponential utility function (Bernoulli functions):

\[ u(x) = 1 - e^{-ax} \]

<table>
<thead>
<tr>
<th>Reward ($)</th>
<th>Utility Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>2000</td>
<td>0.2</td>
</tr>
<tr>
<td>4000</td>
<td>0.4</td>
</tr>
<tr>
<td>6000</td>
<td>0.6</td>
</tr>
<tr>
<td>8000</td>
<td>0.8</td>
</tr>
<tr>
<td>10000</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Risk-Averse Decision Making and Control
Need to Quantify Risk

- Mitigating risk is expensive, how much is it worth?
- **Expected utility theory:**

\[
\mathbb{E}[u(X)] = \mathbb{E}[^\text{utility}(X)]
\]
Need to Quantify Risk

▶ Mitigating risk is expensive, how much is it worth?
▶ Expected utility theory:

$$\mathbb{E}[u(X)] = \mathbb{E}[\text{utility}(X)]$$

▶ Exponential utility function (Bernoulli functions):

$$u(x) = \frac{1 - e^{-ax}}{a}$$
### Example: Buying Car Insurance

**Car value:** $10,000

<table>
<thead>
<tr>
<th>Insurance options</th>
<th>Option</th>
<th>Deductible</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$X_1$</td>
<td>$10,000$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td>$X_2$</td>
<td>$2,000$</td>
<td>$112$</td>
</tr>
<tr>
<td></td>
<td>$X_3$</td>
<td>$100$</td>
<td>$322$</td>
</tr>
</tbody>
</table>

### Expected Utility

<table>
<thead>
<tr>
<th>Event</th>
<th>P</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$112$</td>
<td>$322$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$2,500$</td>
<td>$2,112$</td>
<td>$422$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$10,000$</td>
<td>$2,112$</td>
<td>$422$</td>
</tr>
</tbody>
</table>

**Risk-neutral choice:** no insurance

**Risk-Averse Decision Making and Control**
Example: Buying Car Insurance

---

**Car value:** $10,000

**Insurance options**

<table>
<thead>
<tr>
<th>Option</th>
<th>Deductible</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>$10,000$</td>
<td>$0$</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$2,000$</td>
<td>$112$</td>
</tr>
<tr>
<td>$X_3$</td>
<td>$100$</td>
<td>$322$</td>
</tr>
</tbody>
</table>

**Expected utility:**

<table>
<thead>
<tr>
<th>Event</th>
<th>P</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>−$112$</td>
<td>−$322$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>−$2,500</td>
<td>−$2,112</td>
<td>−$422</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>−$10,000</td>
<td>−$2,112</td>
<td>−$422</td>
</tr>
</tbody>
</table>
### Example: Buying Car Insurance

**Car value:** $10,000

<table>
<thead>
<tr>
<th>Option</th>
<th>Deductible</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>$10,000</td>
<td>$0</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$2,000</td>
<td>$112</td>
</tr>
<tr>
<td>$X_3$</td>
<td>$100</td>
<td>$322</td>
</tr>
</tbody>
</table>

#### Expected utility:

<table>
<thead>
<tr>
<th>Event</th>
<th>$\mathbb{P}$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0</td>
<td>$-112</td>
<td>$-322</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2,500</td>
<td>$-2,112</td>
<td>$-422</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10,000</td>
<td>$-2,112</td>
<td>$-422</td>
</tr>
</tbody>
</table>
**Example: Buying Car Insurance**

**Car value:** $10,000

**Insurance options**

<table>
<thead>
<tr>
<th>Option</th>
<th>Deductible</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1$</td>
<td>$10,000$</td>
<td>$0$</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$2,000$</td>
<td>$112$</td>
</tr>
<tr>
<td>$X_3$</td>
<td>$100$</td>
<td>$322$</td>
</tr>
</tbody>
</table>

**Expected utility:**

<table>
<thead>
<tr>
<th>Event</th>
<th>$P$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$-112$</td>
<td>$-322$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2,500$</td>
<td>$-2,112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10,000$</td>
<td>$-2,112$</td>
<td>$-422$</td>
</tr>
</tbody>
</table>
Example: Buying Car Insurance

Car value: $10,000

<table>
<thead>
<tr>
<th>Insurance options</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option</td>
</tr>
<tr>
<td>$X_1$</td>
</tr>
<tr>
<td>$X_2$</td>
</tr>
<tr>
<td>$X_3$</td>
</tr>
</tbody>
</table>

Expected utility:

<table>
<thead>
<tr>
<th>Event</th>
<th>$P$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$-112$</td>
<td>$-322$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2,500$</td>
<td>$-2,112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10,000$</td>
<td>$-2,112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>$E$</td>
<td>$-237.50$</td>
<td>$-272.00$</td>
<td>$-330.00$</td>
<td></td>
</tr>
</tbody>
</table>

Risk-neutral choice: no insurance
Risk Averse Utility Functions

- **Exponential utility function**

\[ u(x) = 1 - \exp\left(-10^{-6} \cdot (x + 10^5)\right) \]

- \( X_1 \) – no insurance
- \( X_2 \) – high deductible insurance

<table>
<thead>
<tr>
<th>Event</th>
<th>( \mathbb{P} )</th>
<th>( X_1 )</th>
<th>( u(X_1) )</th>
<th>( X_2 )</th>
<th>( u(X_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0</td>
<td>1 111</td>
<td>$-112</td>
<td>1 111</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2500</td>
<td>1 109</td>
<td>$-2112</td>
<td>1 110</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10000</td>
<td>0</td>
<td>$-2112</td>
<td>1 110</td>
</tr>
<tr>
<td><strong>E</strong></td>
<td></td>
<td>$-237.50</td>
<td>1 105</td>
<td>$-272.00</td>
<td>1 111</td>
</tr>
</tbody>
</table>
Risk Averse Utility Functions

- Exponential utility function
  
  \[
  u(x) = \frac{1 - \exp(-10^{-6} \cdot (x + 10^5))}{10^{-6}}
  \]

- \(X_1\) – no insurance
- \(X_2\) – high deductible insurance

<table>
<thead>
<tr>
<th>Event</th>
<th>(\mathbb{P})</th>
<th>(X_1)</th>
<th>(u(X_1))</th>
<th>(X_2)</th>
<th>(u(X_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0</td>
<td>1111</td>
<td>-$112</td>
<td>1111</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>-$2500</td>
<td>1109</td>
<td>-$2112</td>
<td>1110</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>-$10000</td>
<td>0</td>
<td>-$2112</td>
<td>1110</td>
</tr>
<tr>
<td>(\mathbb{E})</td>
<td></td>
<td>-$237.50</td>
<td>1105</td>
<td>-$272.00</td>
<td>1111</td>
</tr>
</tbody>
</table>

Prefer insurance, but difficult to interpret and elicit
Drawbacks of Expected Utility Theory

1. Does not explain human behavior
2. Difficult to elicit utilities
3. Complicates optimization

(Schoemaker 1980)

(Friedman et al. 2014)
Major Alternatives for Measuring Risk

1. **Markowitz portfolios**: Penalize dispersion risk

$$\min_{c \geq 0} \text{Var} \left[ \sum_i c_i \cdot X_i \right]$$

s.t. $$\mathbb{E} \left[ \sum_i c_i \cdot X_i \right] = \mu, \quad \sum_i c_i = 1$$

Limited modeling capability and also penalizes upside
Major Alternatives for Measuring Risk

1. **Markowitz portfolios**: Penalize dispersion risk

\[
\begin{align*}
&\min_{c \geq 0} \quad \text{Var} \left[ \sum_i c_i \cdot X_i \right] \\
&\text{s.t.} \quad E \left[ \sum_i c_i \cdot X_i \right] = \mu, \quad \sum_i c_i = 1
\end{align*}
\]

Limited modeling capability and also penalizes upside

2. **Risk measures**: (Artzner et al. 1999)
   - Value at risk (V@R)
   - Conditional value at risk (CV@R)
   - Coherent measures of risk
Coherent Measures of Risk

Topic of this tutorial

- Alternative to expected utility theory
Coherent Measures of Risk

Topic of this tutorial

- Alternative to expected utility theory
- Flexible modeling framework
Coherent Measures of Risk

Topic of this tutorial

- Alternative to expected utility theory
- Flexible modeling framework
- Convenient to use with optimization and decision making
Coherent Measures of Risk

Topic of this tutorial

- Alternative to expected utility theory
- Flexible modeling framework
- Convenient to use with optimization and decision making
- Easier to elicit than utilities
Coherent Measures of Risk

Topic of this tutorial

- Alternative to expected utility theory
- Flexible modeling framework
- Convenient to use with optimization and decision making
- Easier to elicit than utilities
- Difficulties in sequential decision making
Outline

Introduction to Risk Averse Modeling

(Average) Value at Risk

Coherent Measures of Risk

Risk Measures in Reinforcement Learning

Time consistency of in reinforcement learning

Summary
### Schedule

<table>
<thead>
<tr>
<th>Time</th>
<th>Session</th>
</tr>
</thead>
<tbody>
<tr>
<td>9:00–9:20</td>
<td>Introduction to risk-averse modeling</td>
</tr>
<tr>
<td>9:20–9:40</td>
<td><strong>Value at Risk and Average Value at Risk</strong></td>
</tr>
<tr>
<td>9:40–9:50</td>
<td><strong>Break</strong></td>
</tr>
<tr>
<td>9:50–10:30</td>
<td>Coherent Measures of Risk: Properties and methods</td>
</tr>
<tr>
<td>10:30–11:00</td>
<td><strong>Coffee break</strong></td>
</tr>
<tr>
<td>11:00–12:30</td>
<td>Risk-averse reinforcement learning</td>
</tr>
<tr>
<td>12:30–12:40</td>
<td><strong>Break</strong></td>
</tr>
<tr>
<td>12:40–12:55</td>
<td>Time consistent measures of risk</td>
</tr>
</tbody>
</table>
Risk Measure

**Risk measure**: function $\rho$ that maps random variable to a real number
Risk Measure

**Risk measure**: function $\rho$ that maps random variable to a real number

- **Expectation** is a risk measure
  
  $$
  \rho(X) = \mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)P(\omega)
  $$

- Risk neutral
Risk Measure

Risk measure: function $\rho$ that maps random variable to a real number

- **Expectation** is a risk measure

  $$\rho(X) = \mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)P(\omega)$$

  - Risk neutral

- **Worst-case** is a risk measure

  $$\rho(X) = \min[X] = \min_{\omega \in \Omega} X(\omega)$$

  - Very risk averse
**V@R: Value at Risk**

\[ \rho(X) = V@R_\alpha(X) = \sup \{ t : \mathbb{P}[X \leq t] < \alpha \} \]

Rewards smaller than \( V@R_\alpha(X) \) with probability at most \( \alpha \)

**Example \( \alpha \) values:**

\( \alpha = 0.5 \)  Median
**V@R: Value at Risk**

\[ \rho(X) = V@R_\alpha(X) = \sup \left\{ t : \mathbb{P}[X \leq t] < \alpha \right\} \]

Rewards smaller than \( V@R_\alpha(X) \) with probability at most \( \alpha \)

**Example \( \alpha \) values:**

- \( \alpha = 0.5 \)  Median
- \( \alpha = 0.3 \)  More conservative
\[ \rho(X) = \text{V@R}_\alpha(X) = \sup \left\{ t : \mathbb{P}[X \leq t] < \alpha \right\} \]

Rewards smaller than \( \text{V@R}_\alpha(X) \) with probability at most \( \alpha \)

Example \( \alpha \) values:
- \( \alpha = 0.5 \) Median
- \( \alpha = 0.3 \) More conservative
- \( \alpha = 0.05 \) Conservative
V@R: Value at Risk

\[ \rho(X) = \text{V@R}_\alpha(X) = \sup \{ t : \mathbb{P}[X \leq t] < \alpha \} \]

Rewards smaller than \( \text{V@R}_\alpha(X) \) with probability at most \( \alpha \)

Example \( \alpha \) values:
- \( \alpha = 0.5 \) Median
- \( \alpha = 0.3 \) More conservative
- \( \alpha = 0.05 \) Conservative
- \( \alpha = 0 \) Worst case
V@R Example 1: Cumulative Distribution Function

\( V@R_{0.05}(X) = -1.7 \)
V@R Example 2: Cumulative Distribution Function

\[ V@R_{0.3}(X) = -0.5 \]
Car Insurance And $\text{V@R}$: 25%

<table>
<thead>
<tr>
<th>Event</th>
<th>$\mathbb{P}$</th>
<th>$X_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2,500$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10,000$</td>
</tr>
</tbody>
</table>

$$V\text{@R}_\alpha(X) = \sup \left\{ t : \mathbb{P}[X \leq t] < \alpha \right\} \quad \alpha = 0.25$$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\mathbb{P}[X \leq t]$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2,600$</td>
<td>0.005</td>
<td>0.25</td>
</tr>
<tr>
<td>$-2,500$</td>
<td>0.008</td>
<td>0.25</td>
</tr>
<tr>
<td>$0$</td>
<td>1.000</td>
<td>0.25</td>
</tr>
</tbody>
</table>
Car Insurance And V@R: 8%

<table>
<thead>
<tr>
<th>Event</th>
<th>$P$</th>
<th>$X_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-$2 500</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-$10 000</td>
</tr>
</tbody>
</table>

\[
V@R_\alpha(X) = \sup \left\{ t : P[X \leq t] < \alpha \right\} \quad \alpha = 0.008
\]

<table>
<thead>
<tr>
<th>$t$</th>
<th>$P[X \leq t]$</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-$2 500</td>
<td>0.005</td>
<td>0.008</td>
</tr>
<tr>
<td>$-$2 400</td>
<td>0.008</td>
<td>0.008</td>
</tr>
</tbody>
</table>
Car Insurance And V@R

- $X_1$: no insurance (high risk)
- $X_2$: high deductible insurance (medium risk)
- $X_3$: low deductible insurance (low risk)

<table>
<thead>
<tr>
<th>Event</th>
<th>$P$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$-112$</td>
<td>$-322$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2500$</td>
<td>$-2112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10000$</td>
<td>$-2112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>$E$</td>
<td></td>
<td>$-238$</td>
<td>$-272$</td>
<td>$-330$</td>
</tr>
</tbody>
</table>
Car Insurance And \( \text{V@R} \)

- \( X_1 \): no insurance (high risk)
- \( X_2 \): high deductible insurance (medium risk)
- \( X_3 \): low deductible insurance (low risk)

<table>
<thead>
<tr>
<th>Event</th>
<th>( P )</th>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0</td>
<td>-$112</td>
<td>-$322</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>-$2500</td>
<td>-$2112</td>
<td>-$422</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>-$10000</td>
<td>-$2112</td>
<td>-$422</td>
</tr>
<tr>
<td>( E )</td>
<td></td>
<td>-$238</td>
<td>-$272</td>
<td>-$330</td>
</tr>
<tr>
<td>( \text{V@R}_{0.25} )</td>
<td></td>
<td>$0</td>
<td>-$112</td>
<td>-$322</td>
</tr>
</tbody>
</table>
Car Insurance And V@R

- $X_1$: no insurance (high risk)
- $X_2$: high deductible insurance (medium risk)
- $X_3$: low deductible insurance (low risk)

<table>
<thead>
<tr>
<th>Event</th>
<th>$P$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$-$112</td>
<td>$-$322</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-$2500</td>
<td>$-$2112</td>
<td>$-$422</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-$10000</td>
<td>$-$2112</td>
<td>$-$422</td>
</tr>
<tr>
<td>$E$</td>
<td></td>
<td>$-$238</td>
<td>$-$272</td>
<td>$-$330</td>
</tr>
<tr>
<td>V@R$_{0.25}$</td>
<td></td>
<td>$0$</td>
<td>$-$112</td>
<td>$-$322</td>
</tr>
<tr>
<td>V@R$_{0.05}$</td>
<td></td>
<td>$-$2500</td>
<td>$-$2112</td>
<td>$-$422</td>
</tr>
</tbody>
</table>

Risk-Averse Decision Making and Control
Properties of V@R

- Preserves affine transformations:
  \[ V@R_{\alpha}(\tau \cdot X + c) = \tau \cdot V@R_{\alpha}(X) + c \]

- Simple and intuitive to model and understand
- Compelling meaning in finance
  - Ignores heavy tails
  - Not convex
Properties of \( V@R \)

+ Preserves affine transformations:

\[
V@R_\alpha (\tau \cdot X + c) = \tau \cdot V@R_\alpha (X) + c
\]

+ Simple and intuitive to model and understand
+ Compelling meaning in finance
  - Ignores heavy tails
  - Not convex

Coherent measures of risk: Preserve \( V@R \) positives and improve negatives (Artzner et al. 1999)
Average Value at Risk

- AKA Conditional Value at Risk and Expected Shortfall
- Popular coherent risk measure $\rho$
- Simple definition for atomless distributions:

$$CV@R_\alpha(X) = \mathbb{E}\left[ X \mid X \leq V@R_\alpha(X) \right]$$

- Recall: $V@R_\alpha(X) = \sup \left\{ t : \mathbb{P}[X \leq t] < \alpha \right\}$
- Convex extension of $V@R$ (Rockafellar and Uryasev 2000)
V@R vs CV@R: Cumulative Distribution Function

\[ V@R_{0.3}(X) = -0.5 \]

\[ CV@R_{0.3}(X) = -1.1 \]
**CV@R vs V@R: Heavy Tails**

A more expensive car?

<table>
<thead>
<tr>
<th>Event</th>
<th>$P$</th>
<th>$X_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
</tr>
<tr>
<td>Minor acc.</td>
<td>7.5%</td>
<td>$-2500$</td>
</tr>
<tr>
<td>Major acc.</td>
<td>0.5%</td>
<td>$-10000$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Event</th>
<th>$P$</th>
<th>$X_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
</tr>
<tr>
<td>Minor acc.</td>
<td>7.5%</td>
<td>$-2500$</td>
</tr>
<tr>
<td>Major acc.</td>
<td>0.5%</td>
<td>$-1000000$</td>
</tr>
</tbody>
</table>
CV@R vs V@R: Heavy Tails

A more expensive car?

<table>
<thead>
<tr>
<th>Event</th>
<th>P</th>
<th>$X_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
</tr>
<tr>
<td>Minor acc.</td>
<td>7.5%</td>
<td>−$2500$</td>
</tr>
<tr>
<td>Major acc.</td>
<td>0.5%</td>
<td>−$10000$</td>
</tr>
<tr>
<td>V@R$_{0.05}$</td>
<td></td>
<td>−$2500$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Event</th>
<th>P</th>
<th>$X_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
</tr>
<tr>
<td>Minor acc.</td>
<td>7.5%</td>
<td>−$2500$</td>
</tr>
<tr>
<td>Major acc.</td>
<td>0.5%</td>
<td>−$1000000$</td>
</tr>
<tr>
<td>V@R$_{0.05}$</td>
<td></td>
<td>−$2500$</td>
</tr>
</tbody>
</table>
CV@R vs V@R: Heavy Tails

A more expensive car?

<table>
<thead>
<tr>
<th>Event</th>
<th>P</th>
<th>$X_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0</td>
</tr>
<tr>
<td>Minor acc.</td>
<td>7.5%</td>
<td>$-2500</td>
</tr>
<tr>
<td>Major acc.</td>
<td>0.5%</td>
<td>$-10000</td>
</tr>
<tr>
<td>V@R_{0.05}</td>
<td></td>
<td>$-2500</td>
</tr>
<tr>
<td>CV@R_{0.05}</td>
<td></td>
<td>$-3250</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Event</th>
<th>P</th>
<th>$X_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0</td>
</tr>
<tr>
<td>Minor acc.</td>
<td>7.5%</td>
<td>$-2500</td>
</tr>
<tr>
<td>Major acc.</td>
<td>0.5%</td>
<td>$-10000</td>
</tr>
<tr>
<td>V@R_{0.05}</td>
<td></td>
<td>$-2500</td>
</tr>
<tr>
<td>CV@R_{0.05}</td>
<td></td>
<td>$-102250</td>
</tr>
</tbody>
</table>

Risk-Averse Decision Making and Control
Financial crisis, Value-at-Risk forecasts and the puzzle of dependency modeling

T. Berger *, M. Missong

University of Bremen, Wilhelm-Heinze-Str. 5, 28359 Bremen, Germany

Abstract

Risk-Averse Decision Making and Control
CV@R vs V@R: Continuity

(Average) Value at Risk

Risk-Averse Decision Making and Control
## Schedule

<table>
<thead>
<tr>
<th>Time</th>
<th>Event</th>
</tr>
</thead>
<tbody>
<tr>
<td>9:00–9:20</td>
<td>Introduction to risk-averse modeling</td>
</tr>
<tr>
<td>9:20–9:40</td>
<td>Value at Risk and Average Value at Risk</td>
</tr>
<tr>
<td><strong>9:40–9:50</strong></td>
<td><strong>Break</strong></td>
</tr>
<tr>
<td>9:50–10:30</td>
<td>Coherent Measures of Risk: Properties and methods</td>
</tr>
<tr>
<td>10:30–11:00</td>
<td>Coffee break</td>
</tr>
<tr>
<td>11:00–12:30</td>
<td>Risk-averse reinforcement learning</td>
</tr>
<tr>
<td>12:30–12:40</td>
<td><strong>Break</strong></td>
</tr>
<tr>
<td>12:40–12:55</td>
<td>Time consistent measures of risk</td>
</tr>
</tbody>
</table>
Outline

Introduction to Risk Averse Modeling

(Average) Value at Risk

Coherent Measures of Risk

Risk Measures in Reinforcement Learning

Time consistency of in reinforcement learning

Summary
<table>
<thead>
<tr>
<th>Time</th>
<th>Session</th>
</tr>
</thead>
<tbody>
<tr>
<td>9:00–9:20</td>
<td>Introduction to risk-averse modeling</td>
</tr>
<tr>
<td>9:20–9:40</td>
<td>Value at Risk and Average Value at Risk</td>
</tr>
<tr>
<td>9:40–9:50</td>
<td>Break</td>
</tr>
<tr>
<td>9:50–10:30</td>
<td>Coherent Measures of Risk</td>
</tr>
<tr>
<td>10:30–11:00</td>
<td>Coffee break</td>
</tr>
<tr>
<td>11:00–12:30</td>
<td>Risk-averse reinforcement learning</td>
</tr>
<tr>
<td>12:30–12:40</td>
<td>Break</td>
</tr>
<tr>
<td>12:40–12:55</td>
<td>Time consistent measures of risk</td>
</tr>
</tbody>
</table>
Coherent Measures of Risk

- Generalize CV@R to allow more general models

- Framework introduced in (Artzner et al. 1999)

- **Coherence**: Requirements for risk measure $\rho$ to satisfy

- Our treatment based on (Shapiro, Dentcheva, and Ruszczynski 2009) and (Follmer and Schied 2011)
Coherence Requirements of Risk Measures

1. **Convexity**: (really concavity for maximization!)

   \[ \rho(t \cdot X + (1 - t) \cdot Y) \geq t \cdot \rho(X) + (1 - t) \cdot \rho(Y) \]

2. **Monotonicity**:

   If \( X \succeq Y \), then \( \rho(X) \geq \rho(Y) \)

3. **Translation equivariance**: For a constant \( a \):

   \[ \rho(X + a) = \rho(X) + a \]

4. **Positive homogeneity**: For \( t > 0 \), then:

   \[ \rho(t \cdot X) = t \cdot \rho(X) \]
Convexity

**Why:** Diversification should decrease risk (and it helps with optimization)

\[ \rho(t \cdot X + (1 - t) \cdot Y) \geq t \cdot \rho(X) + (1 - t) \cdot \rho(Y) \]
## Convexity

**Why:** Diversification should decrease risk (and it helps with optimization)

$$\rho(t \cdot X + (1 - t) \cdot Y) \geq t \cdot \rho(X) + (1 - t) \cdot \rho(Y)$$

<table>
<thead>
<tr>
<th>Event</th>
<th>$P$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$\frac{1}{2}X_1 + \frac{1}{2}X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$-112$</td>
<td>$-56$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2500$</td>
<td>$-2112$</td>
<td>$-2306$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10000$</td>
<td>$-2112$</td>
<td>$-6056$</td>
</tr>
<tr>
<td>CV@R</td>
<td></td>
<td>$-238$</td>
<td>$-272$</td>
<td>$-240$</td>
</tr>
</tbody>
</table>
**Convexity**

**Why:** Diversification should decrease risk (and it helps with optimization)

\[
\rho(t \cdot X + (1 - t) \cdot Y) \geq t \cdot \rho(X) + (1 - t) \cdot \rho(Y)
\]

<table>
<thead>
<tr>
<th>Event</th>
<th>(P)</th>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(\frac{1}{2}X_1 + \frac{1}{2}X_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$-112$</td>
<td>$-56$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2500$</td>
<td>$-2112$</td>
<td>$-2306$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10000$</td>
<td>$-2112$</td>
<td>$-6056$</td>
</tr>
<tr>
<td>CV@R</td>
<td></td>
<td>$-238$</td>
<td>$-272$</td>
<td>$-240$</td>
</tr>
</tbody>
</table>

\[-240 \geq \frac{-238 + -272}{2} = -255\]
Monotonicity

**Why:** Do not prefer an outcome that is always worse

\[ X \succeq Y \text{, then } \rho(X) \geq \rho(Y) \]
Monotonicity

Why: Do not prefer an outcome that is always worse

If $X \succeq Y$, then $\rho(X) \geq \rho(Y)$

$X'_2$: Insurance with deductible of $10\,000$

<table>
<thead>
<tr>
<th>Event</th>
<th>$\mathbb{P}$</th>
<th>$X_1$</th>
<th>$X'_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$-112$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2500$</td>
<td>$-2500$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10000$</td>
<td>$-10000$</td>
</tr>
<tr>
<td>$\rho$</td>
<td></td>
<td>$-238$</td>
<td>$-320$</td>
</tr>
</tbody>
</table>
Monotonicity

Why: Do not prefer an outcome that is always worse

If $X \succeq Y$, then $\rho(X) \geq \rho(Y)$

$X'_2$: Insurance with deductible of $10,000$

<table>
<thead>
<tr>
<th>Event</th>
<th>$P$</th>
<th>$X_1$</th>
<th>$X'_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$-112$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2,500$</td>
<td>$-2,500$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10,000$</td>
<td>$-10,000$</td>
</tr>
<tr>
<td>$\rho$</td>
<td></td>
<td>$-238$</td>
<td>$-320$</td>
</tr>
</tbody>
</table>

$-320 \leq -238$
Translation equivariance

**Why:** Risk is measured in the same units as the reward

\[ \rho(X + a) = \rho(X) + a \]
Translation equivariance

Why: Risk is measured in the same units as the reward

\[ \rho(X + a) = \rho(X) + a \]

More expensive insurance by $100

<table>
<thead>
<tr>
<th>Event</th>
<th>( \mathbb{P} )</th>
<th>( X_2 )</th>
<th>( X_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>(-$112)</td>
<td>(-$212)</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>(-$2112)</td>
<td>(-$2212)</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>(-$2112)</td>
<td>(-$2212)</td>
</tr>
<tr>
<td>( \rho )</td>
<td></td>
<td>(-$272)</td>
<td>(-$372)</td>
</tr>
</tbody>
</table>
Translation equivariance

**Why:** Risk is measured in the same units as the reward

\[ \rho(X + a) = \rho(X) + a \]

### More expensive insurance by $100

<table>
<thead>
<tr>
<th>Event</th>
<th>( P )</th>
<th>( X_2 )</th>
<th>( X_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>(-112)</td>
<td>(-212)</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>(-2112)</td>
<td>(-2212)</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>(-2112)</td>
<td>(-2212)</td>
</tr>
<tr>
<td>( \rho )</td>
<td></td>
<td>(-272)</td>
<td>(-372)</td>
</tr>
</tbody>
</table>

\[-372 = -272 - 100\]
Positive homogeneity

**Why:** Risk is measured in the same units as the reward

\[ \rho(t \cdot X) = t \cdot \rho(X) \]
Coherent Measures of Risk

Positive homogeneity

**Why:** Risk is measured in the same units as the reward

\[ \rho(t \cdot X) = t \cdot \rho(X) \]

What if the prices are in €: $1 = €0.94

<table>
<thead>
<tr>
<th>Event</th>
<th>( P )</th>
<th>( X_2 )</th>
<th>( X_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$112</td>
<td>€105</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$2112</td>
<td>€1985</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$2112</td>
<td>€1985</td>
</tr>
<tr>
<td>( \rho )</td>
<td></td>
<td>$272</td>
<td>€256</td>
</tr>
</tbody>
</table>
Positive homogeneity

**Why:** Risk is measured in the same units as the reward

$$\rho(t \cdot X) = t \cdot \rho(X)$$

What if the prices are in €: $1 = €0.94$

<table>
<thead>
<tr>
<th>Event</th>
<th>$\mathbb{P}$</th>
<th>$X_2$</th>
<th>$X_2^€$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>−$112$</td>
<td>−€105</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>−$2112$</td>
<td>−€1985</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>−$2112$</td>
<td>−€1985</td>
</tr>
<tr>
<td>$\rho$</td>
<td></td>
<td>−$272$</td>
<td>−€256</td>
</tr>
</tbody>
</table>

$$−$272 = −€256$$
Convex Risk Measures

Weaker definition than coherent risk measures

1. Convexity:

\[ \rho(t \cdot X + (1 - t) \cdot Y) \leq t \cdot \rho(X) + (1 - t) \cdot \rho(Y) \]

2. Monotonicity:

If \( X \succeq Y \), then \( \rho(X) \geq \rho(Y) \)

3. Translation equivariance: For a constant \( a \):

\[ \rho(X + a) = \rho(X) + a \]

4. Positive homogeneity
Additional Property: Law Invariance

Value of risk measure is independent of the names of the events

Consider a coin flip

<table>
<thead>
<tr>
<th>Event</th>
<th>( \mathbb{P} )</th>
<th>( X )</th>
<th>( Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heads</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Tails</td>
<td>( \frac{1}{2} )</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Require that \( \rho(X) = \rho(Y) \); violated by some coherent risk measures

Distortion risk measures: coherence & law invariance & comonotonicity
Simple Coherent Measures of Risk

- **Expectation:**

\[ \rho(x) = \mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)P(\omega) \]

1. **Convexity:** \( \mathbb{E}[X] \) is linear
2. **Monotonicity:** \( \mathbb{E}[X] \geq \mathbb{E}[Y] \) if \( X \succeq Y \)
3. **Translation equivariance:** \( \mathbb{E}[X + a] = \mathbb{E}[X] + a \)
4. **Positive homogeneity:** \( \mathbb{E}[t \cdot X] = t \cdot \mathbb{E}[X] \) for \( t > 0 \)
Simple Coherent Measures of Risk

- **Expectation:**

  \[ \rho(x) = \mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)P(\omega) \]

  1. **Convexity:** \( \mathbb{E}[X] \) is linear
  2. **Monotonicity:** \( \mathbb{E}[X] \geq \mathbb{E}[Y] \) if \( X \succeq Y \)
  3. **Translation equivariance:** \( \mathbb{E}[X + a] = \mathbb{E}[X] + a \)
  4. **Positive homogeneity:** \( \mathbb{E}[t \cdot X] = t \cdot \mathbb{E}[X] \) for \( t > 0 \)

- **Worst case:**

  \[ \rho(X) = \min[X] = \min_{\omega \in \Omega} X(\omega) \]

  1. **Convexity:** \( \min[X] \) is convex
  2. **Monotonicity:** \( \min[X] \geq \min[Y] \) if \( X \succeq Y \)
  3. **Translation equivariance:** \( \min[X + a] = \min[X] + a \)
  4. **Positive homogeneity:** \( \min[t \cdot X] = t \cdot \mathbb{E}[X] \) for \( t > 0 \)
CV@R for Discrete Distributions

▶ Simple definition is not coherent

\[ CV@R_\alpha(X) = \mathbb{E}\left[ X \middle| X \leq V@R_\alpha(X) \right] \]

▶ Violates convexity when distribution has atoms (discrete distributions)
CV@R for Discrete Distributions

- Simple definition is **not coherent**

\[
CV@R_\alpha(X) = \mathbb{E}\left[ X \mid X \leq V@R_\alpha(X) \right]
\]

- Violates **convexity** when distribution has atoms (discrete distributions)

- **Coherent definition of CV@R:**

\[
CV@R_\alpha(X) = \sup_t \left\{ t + \frac{1}{\alpha} \mathbb{E}[X - t]_+ \right\}
\]

- \( t^* = V@R_\alpha(X) \) when the distribution is atom-less
**CV@R for Discrete Distributions**

- Simple definition is **not coherent**

\[
CV@R_\alpha(X) = \mathbb{E}\left[ X \mid X \leq V@R_\alpha(X) \right]
\]

- Violates **convexity** when distribution has atoms (discrete distributions)

- **Coherent definition of CV@R:**

\[
CV@R_\alpha(X) = \sup_t \left\{ t + \frac{1}{\alpha} \mathbb{E}[X - t]_- \right\}
\]

- \( t^* = V@R_\alpha(X) \) when the distribution is atom-less

- Definitions the same for continuous distributions

---

Risk-Averse Decision Making and Control
Computing CV@R

- **Discrete distributions**: Solve a linear program

\[
\begin{align*}
\max_{t,y} & \quad t + \frac{1}{\alpha} p^\top y \\
\text{s.t.} & \quad y \leq X - t, \\
& \quad y \leq 0
\end{align*}
\]

- **Continuous distributions**: Closed form for many (Nadarajah, Zhang, and Chan 2014; Andreev, Kanto, and Malo 2005)
Car Insurance and CV@R

- $X_1$ – no insurance
- $X_2$ – high deductible insurance
- $X_3$ – low deductible insurance

<table>
<thead>
<tr>
<th>Event</th>
<th>$\mathbb{P}$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$-112$</td>
<td>$-322$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2500$</td>
<td>$-2112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10000$</td>
<td>$-2112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>$\mathbb{E}$</td>
<td></td>
<td>$-238$</td>
<td>$-272$</td>
<td>$-330$</td>
</tr>
</tbody>
</table>
Coherent Measures of Risk

## Car Insurance and CV@R

- $X_1$ – no insurance
- $X_2$ – high deductible insurance
- $X_3$ – low deductible insurance

<table>
<thead>
<tr>
<th>Event</th>
<th>$\mathbb{P}$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$-112$</td>
<td>$-322$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2500$</td>
<td>$-2112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10000$</td>
<td>$-2112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>$\mathbb{E}$</td>
<td></td>
<td>$-238$</td>
<td>$-272$</td>
<td>$-330$</td>
</tr>
<tr>
<td>$\text{V@R}_{0.25}$</td>
<td></td>
<td>$0$</td>
<td>$-112$</td>
<td>$-322$</td>
</tr>
<tr>
<td>$\text{CV@R}_{0.25}$</td>
<td></td>
<td>$-950$</td>
<td>$-752$</td>
<td>$-354$</td>
</tr>
</tbody>
</table>

Risk-Averse Decision Making and Control
## Car Insurance and CV@R

- $X_1$ – no insurance
- $X_2$ – high deductible insurance
- $X_3$ – low deductible insurance

<table>
<thead>
<tr>
<th>Event</th>
<th>$\mathbb{P}$</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No accident</td>
<td>92%</td>
<td>$0$</td>
<td>$-112$</td>
<td>$-322$</td>
</tr>
<tr>
<td>Minor accident</td>
<td>7.5%</td>
<td>$-2500$</td>
<td>$-2112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>Major accident</td>
<td>0.5%</td>
<td>$-10000$</td>
<td>$-2112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>$\mathbb{E}$</td>
<td></td>
<td>$-238$</td>
<td>$-272$</td>
<td>$-330$</td>
</tr>
<tr>
<td>$\mathbb{V@R}_{0.25}$</td>
<td></td>
<td>$0$</td>
<td>$-112$</td>
<td>$-322$</td>
</tr>
<tr>
<td>$\mathbb{CV@R}_{0.25}$</td>
<td></td>
<td>$-950$</td>
<td>$-752$</td>
<td>$-354$</td>
</tr>
<tr>
<td>$\mathbb{V@R}_{0.05}$</td>
<td></td>
<td>$-2500$</td>
<td>$-2112$</td>
<td>$-422$</td>
</tr>
<tr>
<td>$\mathbb{CV@R}_{0.05}$</td>
<td></td>
<td>$-3250$</td>
<td>$-2112$</td>
<td>$-422$</td>
</tr>
</tbody>
</table>
Robust Representation of Coherent Risk Measures

- **Important representation for analysis and optimization**
- For any coherent risk measure $\rho$:

\[
\rho(X) = \min_{\xi \in \mathcal{A}} \mathbb{E}[X] = \inf_{\xi \in \mathcal{A}} \xi^\top X
\]
Robust Representation of Coherent Risk Measures

- **Important representation for analysis and optimization**
- For any coherent risk measure $\rho$:

  $$\rho(X) = \min_{\xi \in \mathcal{A}} \mathbb{E}_\xi [X] = \inf_{\xi \in \mathcal{A}} \xi^\top X$$

- $\mathcal{A}$ is a set of measures such that is:
  1. convex
  2. bounded
  3. closed
Robust Representation of Coherent Risk Measures

- Important representation for analysis and optimization
- For any coherent risk measure \( \rho \):
  \[
  \rho(X) = \min_{\xi \in \mathcal{A}} \mathbb{E}_{\xi}[X] = \inf_{\xi \in \mathcal{A}} \xi^\top X
  \]

- \( \mathcal{A} \) is a set of measures such that is:
  1. convex
  2. bounded
  3. closed

- Proof: Double convex conjugate
  - Convex conjugate:
    \[
    \rho^*(y) = \sup_{x} x^\top y - \rho(x)
    \]
  - Fenchel–Moreau theorem:
    \[
    \rho^{**}(x) = \rho(x)
    \]
Coherent Measures of Risk

Robust Set for \(CV@R\)

\[
CV@R_\alpha(X) = \sup_t \left\{ t + \frac{1}{\alpha} \mathbb{E}[X - t]_-ight\}
\]

- Robust representation:

\[
\rho(X) = \inf_{\xi \in \mathcal{A}} \mathbb{E}_{\xi}[X]
\]

- Robust set for probability distribution \(P\):

\[
\mathcal{A} = \left\{ \xi \geq 0 \ | \ \xi \leq \frac{1}{\alpha} P, \ 1^\top \xi = 1 \right\}
\]

Risk-Averse Decision Making and Control
Robust Set for CV@R

- Robust representation:
  \[ \rho(X) = \min_{\xi \in \mathcal{A}} \mathbb{E}_\xi [X] \]
  \[ \mathcal{A} = \left\{ \xi \geq 0 \mid \xi \leq \frac{1}{\alpha} P, \ 1^\top \xi = 1 \right\} \]

- Random variable: \( X = [10, 5, 2] \)
- Probability distribution: \( p = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}] \)
- CV@R\(_{1/2}(X) = \)
  \[ \min_{\xi \geq 0} \ 10 \xi_1 + 5 \xi_2 + 2 \xi_3 \]
  \[ \xi_i \leq \frac{1}{\alpha} p_i = \frac{1}{2} \frac{1}{1/3} = \frac{2}{3} \quad \xi_1 + \xi_2 + \xi_3 = 1 \]
Other Coherent Risk Measures

1. Combination of expectation and $CV_{\alpha}$

2. Entropic risk measure

3. Coherent entropic risk measure (convex, incoherent)

4. Risk measures from utility functions

5. ...
Convex Combination of Expectation and CV@R

- CV@R ignores the mean return
- Risk-averse solutions bad in expectation
- Practical trade-off: Combine mean and risk

\[ \rho(X) = c \cdot \mathbb{E}[X] + (1 - c) \cdot \text{CV@R}_\alpha(X) \]
Entropic Risk Measure

\[ \rho(X) = -\frac{1}{\tau} \ln \mathbb{E}[e^{-\tau \cdot X}] \quad \tau > 0 \]

- Convex risk measure
Entropic Risk Measure

\[ \rho(X) = -\frac{1}{\tau} \ln \mathbb{E}[e^{-\tau \cdot X}] \quad \tau > 0 \]

- Convex risk measure
- Incoherent (violates translation invariance)
- No robust representation
Entropic Risk Measure

\[ \rho(X) = -\frac{1}{\tau} \ln \mathbb{E}[e^{-\tau \cdot X}] \quad \tau > 0 \]

- Convex risk measure
- Incoherent (violates translation invariance)
- No robust representation
- **Coherent entropic risk measure:** (Föllmer and Knispel 2011)

\[ \rho(X) = \max_{\xi \geq 0} \left\{ \mathbb{E}_\xi[X] \mid KL(\xi \mid P) \leq c, 1^\top \xi = 1 \right\} \]
Risk Measure From Utility Function

- Concave utility function $u(\cdot)$
- Construct a **coherent** risk measure from $g$?
Risk Measure From Utility Function

- Concave utility function \( u(\cdot) \)
- Construct a **coherent** risk measure from \( g \)?
- **Direct construction:**

\[
\rho(X) = \mathbb{E}[u(X)]
\]

Not coherent or convex
Coherent Measures of Risk

Risk Measure From Utility Function

- Concave utility function $u(\cdot)$
- Construct a coherent risk measure from $g$?
- **Direct construction:**

$$\rho(X) = \mathbb{E}[u(X)]$$

Not coherent or convex

- **Optimized Certainty Equivalent** (Ben-Tal and Teboulle 2007)

$$\rho(X) = \sup_t (t + \mathbb{E}[g(X - t)])$$
Optimized Certainty Equivalent

\[ \rho(X) = \sup_t (t + \mathbb{E}[g(X - t)]) \]

- How much consume now given uncertain future
Optimized Certainty Equivalent

\[ \rho(X) = \sup_t (t + \mathbb{E}[g(X - t)]) \]

- How much consume now given uncertain future
- Convex risk measure for any concave \( u \)
- Coherent risk measure for pos. homogeneous \( u \)
Optimized Certainty Equivalent

\[ \rho(X) = \sup_t (t + \mathbb{E}[g(X - t)]) \]

- How much consume now given uncertain future
- Convex risk measure for any concave \( u \)
- Coherent risk measure for pos. homogeneous \( u \)
- Exponential \( u \): OCE = entropic risk measure
- Piecewise linear \( u \): OCE = CV@R
Recommended References

- Lectures on Stochastic Programming: Modeling and Theory (Shapiro, Dentcheva, and Ruszczynski 2014)

- Stochastic Finance: An Introduction in Discrete Time (Follmer and Schied 2011)
Remainder of Tutorial: Multistage Optimization

- How to apply risk measures when optimizing over multiple time steps

- Results in machine learning and reinforcement learning

- Time or dynamic consistency in multiple time steps
## Schedule

<table>
<thead>
<tr>
<th>Time</th>
<th>Session</th>
</tr>
</thead>
<tbody>
<tr>
<td>9:00–9:20</td>
<td>Introduction to risk-averse modeling</td>
</tr>
<tr>
<td>9:20–9:40</td>
<td>Value at Risk and Average Value at Risk</td>
</tr>
<tr>
<td>9:40–9:50</td>
<td>Break</td>
</tr>
<tr>
<td>9:50–10:30</td>
<td>Coherent Measures of Risk: Properties and methods</td>
</tr>
<tr>
<td><strong>10:30–11:00</strong></td>
<td><strong>Coffee break</strong></td>
</tr>
<tr>
<td>11:00–12:30</td>
<td>Risk-averse reinforcement learning</td>
</tr>
<tr>
<td>12:30–12:40</td>
<td>Break</td>
</tr>
<tr>
<td>12:40–12:55</td>
<td>Time consistent measures of risk</td>
</tr>
</tbody>
</table>
Outline

Introduction to Risk Averse Modeling

(Average) Value at Risk

Coherent Measures of Risk

Risk Measures in Reinforcement Learning

Time consistency of in reinforcement learning

Summary
## Schedule

<table>
<thead>
<tr>
<th>Time</th>
<th>Session</th>
</tr>
</thead>
<tbody>
<tr>
<td>9:00–9:20</td>
<td>Introduction to risk-averse modeling</td>
</tr>
<tr>
<td>9:20–9:40</td>
<td>Value at Risk and Average Value at Risk</td>
</tr>
<tr>
<td>9:40–9:50</td>
<td>Break</td>
</tr>
<tr>
<td>9:50–10:30</td>
<td>Coherent Measures of Risk: Properties and methods</td>
</tr>
<tr>
<td>10:30–11:00</td>
<td>Coffee break</td>
</tr>
<tr>
<td><strong>11:00–12:30</strong></td>
<td><strong>Risk-averse reinforcement learning</strong></td>
</tr>
<tr>
<td>12:30–12:40</td>
<td>Break</td>
</tr>
<tr>
<td>12:40–12:55</td>
<td>Time consistent measures of risk</td>
</tr>
</tbody>
</table>
Please see the other slide deck
## Schedule

<table>
<thead>
<tr>
<th>Time</th>
<th>Session</th>
</tr>
</thead>
<tbody>
<tr>
<td>9:00–9:20</td>
<td>Introduction to risk-averse modeling</td>
</tr>
<tr>
<td>9:20–9:40</td>
<td>Value at Risk and Average Value at Risk</td>
</tr>
<tr>
<td>9:40–9:50</td>
<td>Break</td>
</tr>
<tr>
<td>9:50–10:30</td>
<td>Coherent Measures of Risk: Properties and methods</td>
</tr>
<tr>
<td>10:30–11:00</td>
<td>Coffee break</td>
</tr>
<tr>
<td>11:00–12:30</td>
<td>Risk-averse reinforcement learning</td>
</tr>
<tr>
<td>12:30–12:40</td>
<td>Break</td>
</tr>
<tr>
<td>12:40–12:55</td>
<td>Time consistent measures of risk</td>
</tr>
</tbody>
</table>
Outline

Introduction to Risk Averse Modeling

(Average) Value at Risk

Coherent Measures of Risk

Risk Measures in Reinforcement Learning

Time consistency of in reinforcement learning

Summary
## Schedule

<table>
<thead>
<tr>
<th>Time</th>
<th>Session</th>
</tr>
</thead>
<tbody>
<tr>
<td>9:00–9:20</td>
<td>Introduction to risk-averse modeling</td>
</tr>
<tr>
<td>9:20–9:40</td>
<td>Value at Risk and Average Value at Risk</td>
</tr>
<tr>
<td>9:40–9:50</td>
<td>Break</td>
</tr>
<tr>
<td>9:50–10:30</td>
<td>Coherent Measures of Risk: Properties and methods</td>
</tr>
<tr>
<td>10:30–11:00</td>
<td>Coffee break</td>
</tr>
<tr>
<td>11:00–12:30</td>
<td>Risk-averse reinforcement learning</td>
</tr>
<tr>
<td>12:30–12:40</td>
<td>Break</td>
</tr>
<tr>
<td><strong>12:40–12:55</strong></td>
<td><strong>Time consistent measures of risk</strong></td>
</tr>
</tbody>
</table>
Example: Driving Test Discount

Option 1: Plain Insurance

- Cost: $9.00
- No deductible
- Certain expected outcome:

\[ \mathbb{E}[X_1] = -9.00 \]

\[ \rho(X_1) = \mathbb{E}[X_1] = -9.00 \]
Example: Driving Test Discount

Option 1: Plain Insurance

- Cost: $9.00
- No deductible
- Certain expected outcome:

\[ \mathbb{E}[X_1] = -9.00 \]

\[ \rho(X_1) = \mathbb{E}[X_1] = -9.00 \]

Option 2: Custom Insurance

- Take a safety exam
- Pass with probability \( \frac{1}{2} \)
  - OK \( [P = \frac{2}{3}] \): +$5.00
  - Not \( [P = \frac{2}{3}] \): −$20.00
- Fail with probability \( \frac{1}{2} \)
  - OK \( [P = \frac{2}{3}] \): −$5.00
  - Not \( [P = \frac{2}{3}] \): −$10.00
Example: Driving Test Discount

Option 1: Plain Insurance

- Cost: $9.00
- No deductible
- Certain expected outcome:
  \[ \mathbb{E}[X_1] = -9.00 \]
  \[ \rho(X_1) = \mathbb{E}[X_1] = -9.00 \]

Option 2: Custom Insurance

- Take a safety exam
  - Pass with probability \( \frac{1}{2} \)
    - OK \( \mathbb{P} = \frac{2}{3} \): +$5.00
    - Not \( \mathbb{P} = \frac{2}{3} \): −$20.00
  - Fail with probability \( \frac{1}{2} \)
    - OK \( \mathbb{P} = \frac{2}{3} \): −$5.00
    - Not \( \mathbb{P} = \frac{2}{3} \): −$10.00

Risk measure: \( \rho = \text{CV@R}_{2/3} \)
Risk Measure of Option 2

Risk measure:
\[ \rho(X_2) = \text{CV@R}_{2/3}(X_2) \]

<table>
<thead>
<tr>
<th>( P )</th>
<th>( X_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1/6 )</td>
<td>( -5 )</td>
</tr>
<tr>
<td>( 1/6 )</td>
<td>( -5 )</td>
</tr>
<tr>
<td>( 1/6 )</td>
<td>( -10 )</td>
</tr>
<tr>
<td>( 1/6 )</td>
<td>( 5 )</td>
</tr>
<tr>
<td>( 1/6 )</td>
<td>( 5 )</td>
</tr>
<tr>
<td>( 1/6 )</td>
<td>( -20 )</td>
</tr>
</tbody>
</table>
Risk Measure of Option 2

Risk measure:

$$\rho(X_2) = \text{CV@R}_{2/3}(X_2)$$

<table>
<thead>
<tr>
<th>$P$</th>
<th>$X_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{6}$</td>
<td>$-5$</td>
</tr>
<tr>
<td>$\frac{1}{6}$</td>
<td>$-5$</td>
</tr>
<tr>
<td>$\frac{1}{6}$</td>
<td>$-10$</td>
</tr>
<tr>
<td>$\frac{1}{6}$</td>
<td>$5$</td>
</tr>
<tr>
<td>$\frac{1}{6}$</td>
<td>$5$</td>
</tr>
<tr>
<td>$\frac{1}{6}$</td>
<td>$-20$</td>
</tr>
</tbody>
</table>

$$\rho(X_2) = \frac{-5 - 5 - 10 - 20}{4} = \frac{-40}{4} = -10.0$$

$$-10.0 < -9.0 = \rho(X_1)$$
Risk Measure of Option 2

Risk measure:
\[ \rho(X_2) = \text{CV@R}_{2/3}(X_2) \]

\[
\begin{array}{c|c}
\mathbb{P} & X_2 \\
\hline
1/6 & -5 \\
1/6 & -5 \\
1/6 & -10 \\
1/6 & 5 \\
1/6 & 5 \\
1/6 & -20 \\
\end{array}
\]

\[ \rho(X_2) < \rho(X_1) \]

Prefer option 1
Optimal Solution of Subproblems

Recall we prefer option 1: $\rho(X_1) = -9$

\[
\begin{array}{ccc}
\text{Pass test} & \text{NOK} & \text{OK} \\
\frac{1}{3} & -20 & 5 \\
\frac{1}{3} & 0 & 5 \\
\text{OK} & 5 & 5 \\
\end{array}
\]

Time consistency of in reinforcement learning

Risk-Averse Decision Making and Control
Optimal Solution of Subproblems

Recall we prefer option 1: $\rho(X_1) = -9$

If pass, prefer option 2

\[
\rho(X_2 \mid \text{Pass}) = \frac{-20 + 5}{2} = -7.5
\]

If pass, prefer option 2
Optimal Solution of Subproblems

Recall we **prefer option 1**: $\rho(X_1) = -9$

<table>
<thead>
<tr>
<th>$P$</th>
<th>1/3</th>
<th>1/3</th>
<th>1/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_2$</td>
<td>-10</td>
<td>-5</td>
<td>-5</td>
</tr>
</tbody>
</table>
Optimal Solution of Subproblems

Recall we prefer option 1: \( \rho(X_1) = -9 \)

<table>
<thead>
<tr>
<th>( P )</th>
<th>1/3</th>
<th>1/3</th>
<th>1/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_2 )</td>
<td>-10</td>
<td>-5</td>
<td>-5</td>
</tr>
</tbody>
</table>
Optimal Solution of Subproblems

Recall we prefer option 1: $\rho(X_1) = -9$

If pass, prefer option 2

If fail, prefer option 2

\[ \rho(X_2 | \text{Fail}) = \frac{-15 + 5}{2} = -7.5 \]
Optimal Solution of Subproblems

Recall we prefer option 1: \( \rho(X_1) = -9 \)

If pass, prefer option 2

\[ \rho(X_2 \mid \text{Pass}) = \frac{-20 + 5}{2} = -7.5 \]

If fail, prefer option 2

\[ \rho(X_2 \mid \text{Fail}) = \frac{-15 + 5}{2} = -7.5 \]

Time inconsistent behavior (Roorda, Schumacher, and Engwerda 2005; Iancu, Petrik, and Subramanian 2015)
Time Consistent Risk Measures

- Filtration (scenario tree) of rewards with $T$ levels:

  $$X_1, X_2, X_3, \ldots, X_T$$

- **Dynamic risk measure** at time $t$:

  $$\rho_t(X_t + \cdots + X_T)$$
Time Consistent Risk Measures

- Filtration (scenario tree) of rewards with $T$ levels:
  \[ X_1, X_2, X_3, \ldots, X_T \]

- Dynamic risk measure at time $t$:
  \[ \rho_t(X_t + \cdots + X_T) \]

- Time consistent: if for all $X, Y$ (also dynamic consistent)
  \[ \rho_{t+1}(X_t + \cdots) \geq \rho_{t+1}(Y_t + \cdots) \Rightarrow \rho_t(X_t + \cdots) \geq \rho_t(Y_t + \cdots) \]
Time Consistent Risk Measures

- Filtration (scenario tree) of rewards with $T$ levels:

$$X_1, X_2, X_3, \ldots, X_T$$

- **Dynamic risk measure** at time $t$:

$$\rho_t(X_t + \cdots + X_T)$$

- **Time consistent**: if for all $X, Y$ (also dynamic consistent)

$$\rho_{t+1}(X_t + \cdots) \geq \rho_{t+1}(Y_t + \cdots) \Rightarrow \rho_t(X_t + \cdots) \geq \rho_t(Y_t + \cdots)$$

- Similar to subproblem optimality in programming optimality
Time Consistency via Iterated Risk Mappings

- **Time consistent** risk measures must be composed of **iterated risk mappings** (Roorda, Schumacher, and Engwerda 2005):
  \[ \mu_1, \mu_2, \ldots, \mu_t \]

- Dynamic risk measure:
  \[ \rho_t(X_t + \cdots + X_T) = \mu_t(X_t + \mu_{t+1}(X_{t+1} + \mu_{t+2}(x_{t+3} + \cdots))) \]

- Each \( \mu_t \): a coherent risk measure applied on subtree of filtration
Time Consistency via Iterated Risk Mappings

- **Time consistent** risk measures must be composed of iterated risk mappings (Roorda, Schumacher, and Engwerda 2005):
  \[ \mu_1, \mu_2, \ldots, \mu_t \]

- Dynamic risk measure:
  \[ \rho_t(X_t + \cdots + X_T) = \mu_t(X_t + \mu_{t+1}(X_{t+1} + \mu_{t+2}(x_{t+3} + \cdots))) \]

- Each \( \mu_t \): a coherent risk measure applied on subtree of filtration

- Markov risk measures for MDPs (Ruszczynski 2010)
Computing Time Consistent Risk Measure

\[ \rho(X_2 \mid \text{Pass}) = \frac{-20 + 5}{2} = -7.5 \]
Computing Time Consistent Risk Measure

\[ \rho(X_2 \mid \text{Fail}) = \frac{-15 + 5}{2} = -7.5 \]
Computing Time Consistent Risk Measure

\[ \rho(X_2) = \rho(-7.5) = -7.5 > -9 \]
Computing Time Consistent Risk Measure

\[ \rho(X_2 \mid \text{Pass}) = \frac{-20 + 5}{2} = -7.5 \]
\[ \rho(X_2 \mid \text{Fail}) = \frac{-15 + 5}{2} = -7.5 \]

\[ \rho(X_2) = \rho(-7.5) = -7.5 > -9 \]

Consistently prefer option 1 throughout the execution
Time consistency of in reinforcement learning

Approximating Inconsistent Risk Measures

- Time consistent risk measures are difficult to specify
- Approximate an inconsistent risk measure by a consistent one?
- **Best lower bound**: e.g. what is the best $\alpha_1, \alpha_2$ such that
  \[
  \text{CV@R}_{\alpha_1}(\text{CV@R}_{\alpha_2}(X)) \leq \text{CV@R}_{\alpha}(X) \text{ for all } X
  \]
- **Best upper bound**: e.g. what is the best $\alpha_1, \alpha_2$ such that
  \[
  \text{CV@R}_{\alpha_1}(\text{CV@R}_{\alpha_2}(X)) \geq \text{CV@R}_{\alpha}(X) \text{ for all } X
  \]

(Iancu, Petrik, and Subramanian 2015)
Best Time Consistent Bounds

- Compare robust sets of consistent and inconsistent measures
- **Main insight**: need to compare *down-monotone* closures of robust sets
Time Consistent Bounds: Main Results

- **Lower consistent bound:**
  - Uniformly tightest bound can be constructed in polynomial time
  - **Method**: rectangularization

- **Upper consistent bound:**
  - NP hard to even **evaluate** how tight the approximation is
  - Approximation can be tighter than the lower bound
Planning with Time Consistent Risk Measures

- Stochastic dual dynamic programming (Shapiro 2012)
- Applied in reinforcement learning (Petrik and Subramanian 2012)
- Only entropic dynamically consistent risk measures are law invariant (Kupper and Schachermayer 2006)
Outline

Introduction to Risk Averse Modeling

(Average) Value at Risk

Coherent Measures of Risk

Risk Measures in Reinforcement Learning

Time consistency of in reinforcement learning

Summary
Risk Measures: Many Other Topics

1. Elicitation of risk measures
2. Estimation of risk measure from samples
3. Relationship to acceptance sets
4. Relationship to robust optimization
Take Home Messages

- Coherent risk measures are a convenient and established risk aversion framework
Take Home Messages

- Coherent risk measures are a convenient and established risk aversion framework
- Computations with coherent risk measure are more efficient than with utility functions
Take Home Messages

- Coherent risk measures are a convenient and established risk aversion framework
- Computations with coherent risk measure are more efficient than with utility functions
- Risk measures ($\text{V@R}$, $\text{CV@R}$) are more intuitive than utility functions
Take Home Messages

- Coherent risk measures are a convenient and established risk aversion framework
- Computations with coherent risk measure are more efficient than with utility functions
- Risk measures ($\text{V@R}, \text{CV@R}$) are more intuitive than utility functions
- Time consistency is important in dynamic settings, but can be difficult to achieve (open research problems)
Take Home Messages

- Coherent risk measures are a convenient and established risk aversion framework
- Computations with coherent risk measure are more efficient than with utility functions
- Risk measures ($\text{V@R}$, $\text{CV@R}$) are more intuitive than utility functions
- Time consistency is important in dynamic settings, but can be difficult to achieve (open research problems)
- Risk measures are making inroads in reinforcement learning and artificial intelligence
Thank you!!
Bibliography I


Bibliography II


Bibliography V


Bibliography VI

