Support Vector Machines

Maximum Margin Classifiers

Marek Petrik

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Classifiers

- Which classifiers do you know? (5+)

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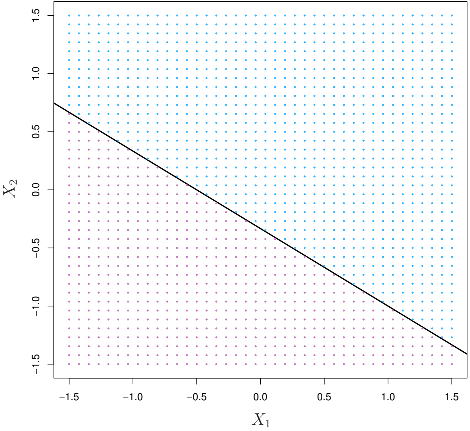
- Which ones are generative/discriminative?

Classifiers

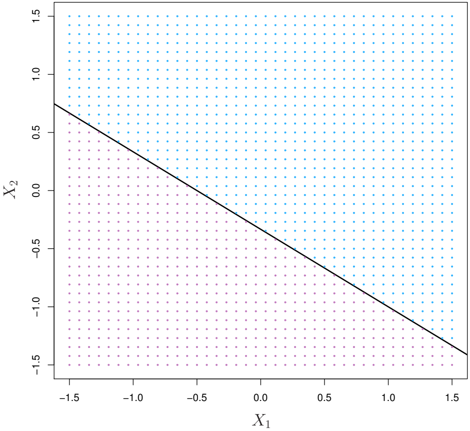
- Which classifiers do you know? (5+)
- Which ones are generative/discriminative?
- Do we need any more? Why?

Separating Hyperplane



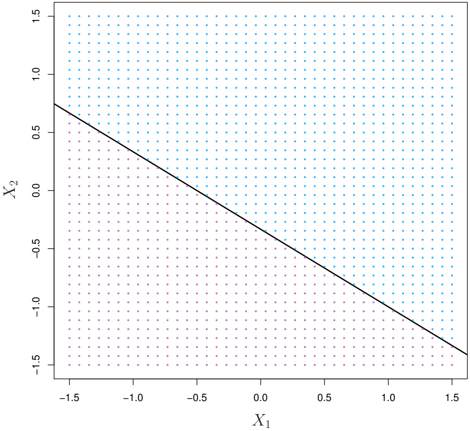
Hyperplane: $\beta_0 + x^\top \beta = 0$

Separating Hyperplane



Blue: $\beta_0 + x^\top \beta > 0$

Separating Hyperplane

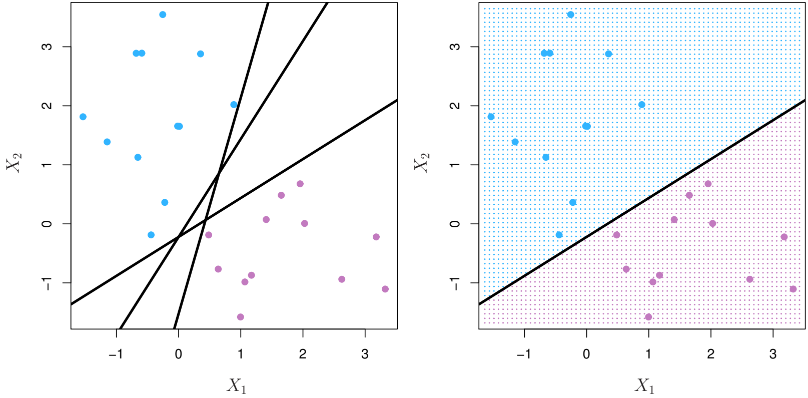


Red: $\beta_0 + x^\top \beta < 0$

Question

- Which other classification methods classify using a separating hyperplane?

Best Separating Hyperplane



- Data is separable
- Why would either one be better than others?

Separating Hyperplane Methods

How is it computed?

- **Logistic regression**:

Separating Hyperplane Methods

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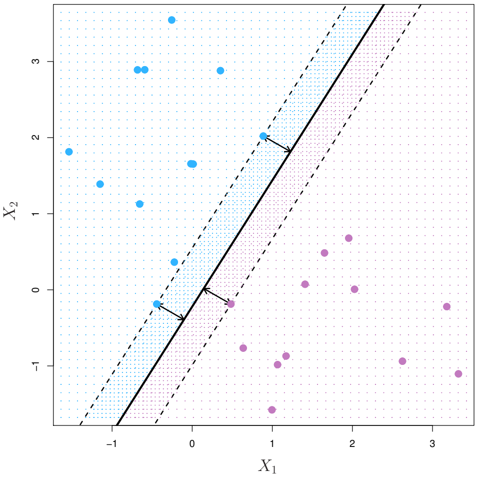
Separating Hyperplane Methods

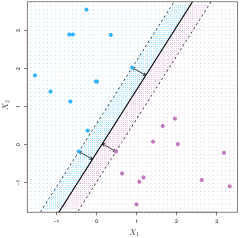
How is it computed?

- **Logistic regression**: Maximum likelihood
- **LDA**: Maximum likelihood
- **Support vector machines**: Maximum margin

Maximum Margin Hyperplane



Computing Maximum Margin Hyperplane

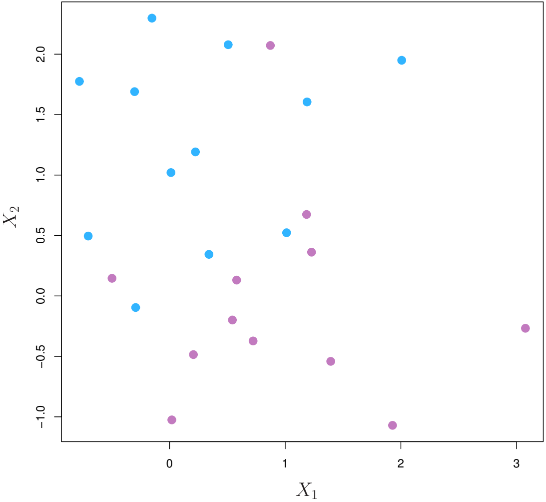


- **Class labels**: $y_i \in \{-1, +1\}$ (not $\{0, 1\}$)
- Solve a **quadratic program** (assume one of the features is a constant to get the equivalent of an intercept)

$$\begin{aligned} \max_{\beta, M} \quad & M \\ \text{s.t.} \quad & y_i(\beta^\top x) \geq M \\ & \|\beta\|_2 = 1 \end{aligned}$$

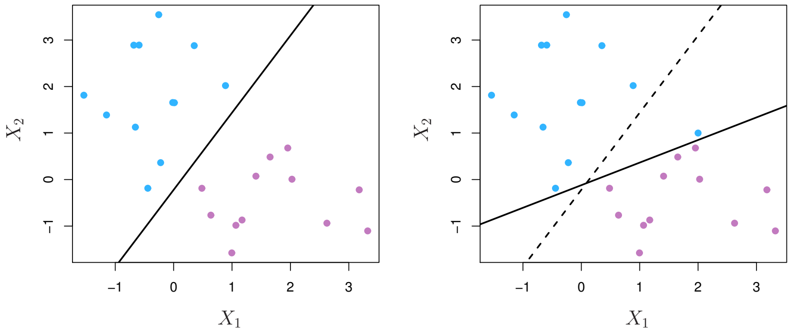
Non-separable Case

- Rarely lucky enough to get separable classes



Almost Unseparable Cases

- Maximum margin can be brittle even when classes are separable



Introducing Slack Variables

- **Maximum margin classifier**

$$\begin{aligned} \max_{\beta,M} \quad & M \\ \text{s.t.} \quad & y_i(\beta^\top x) \geq M \\ & \|\beta\|_2 = 1 \end{aligned}$$

- **Support Vector Classifier** a.k.a Linear SVM

$$\begin{aligned} \max_{\beta,M,\epsilon\geq 0} \quad & M \\ \text{s.t.} \quad & y_i(\beta^\top x) \geq (M - \epsilon_i) \\ & \|\beta\|_2 = 1 \\ & \|\epsilon\|_1 \leq C \end{aligned}$$

- Slack variables: ϵ
- Parameter: C

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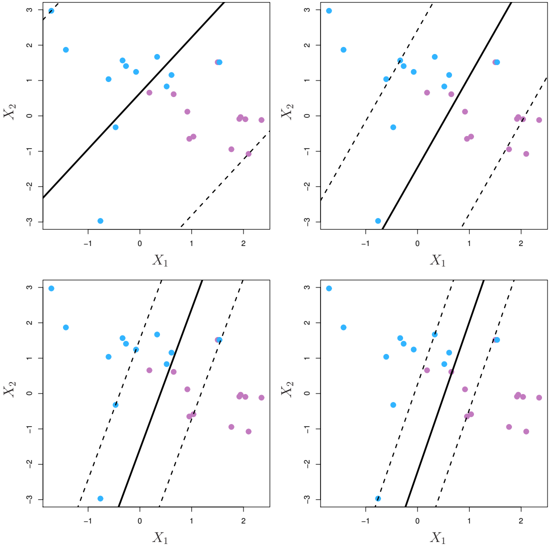
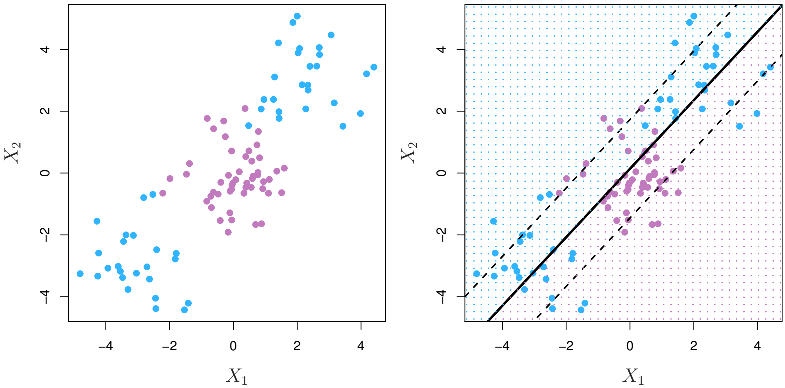
- Slack variables: ϵ
- Parameter: C What if $C = 0$?

Effect of Decreasing Parameter C



What About Nonlinearity?



Dealing with Nonlinearity

- Introduce more features, just like with logistic regression
- It is possible to do better with SVMs
- **Primal Quadratic Program**

$$
\begin{aligned}
\max_{\beta, M} \quad & M \\
\text{s.t.} \quad & y_i(\beta^\top x) \geq M \\
& \|\beta\|_2 = 1
\end{aligned}
$$

- Equivalent **Dual Quadratic Program** (usually max-min, not here)

$$
\begin{aligned}
\max_{\alpha \geq 0} \quad & \sum_{l=1}^{M} \alpha_l - \frac{1}{2} \sum_{j,k=1}^{M} \alpha_j \alpha_k y_j y_k \langle x_j, x_k \rangle \\
\text{s.t.} \quad & \sum_{l=1}^{M} \alpha_l y_l = 0
\end{aligned}
$$

SVM Dual Representation

- **Dual Quadratic Program** (usually max-min, not here)

$$\max_{\alpha \geq 0} \quad \sum_{l=1}^{M} \alpha_l - \frac{1}{2} \sum_{j,k=1}^{M} \alpha_j \alpha_k y_j y_k \langle x_j, x_k \rangle$$

$$\text{s.t.} \quad \sum_{l=1}^{M} \alpha_l y_l = 0$$

- **Representer theorem**: (classification test):

$$f(z) = \sum_{l=1}^{M} \alpha_l y_l \langle z, x_l \rangle > 0$$

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- Only need the inner product between data points
- Define a **kernel function** by projecting data to higher dimensions:

$$k(x_1, x_2) = \langle \phi(x_1), \phi(x_2) \rangle$$

Kernelized SVM

- **Dual Quadratic Program** (usually max-min, not here)

$$\max_{\alpha \geq 0} \quad \sum_{l=1}^{M} \alpha_l - \frac{1}{2} \sum_{j,k=1}^{M} \alpha_j \alpha_k y_j y_k k(x_j, x_k)$$

$$\text{s.t.} \quad \sum_{l=1}^{M} \alpha_l y_l = 0$$

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Kernels

- Polynomial kernel

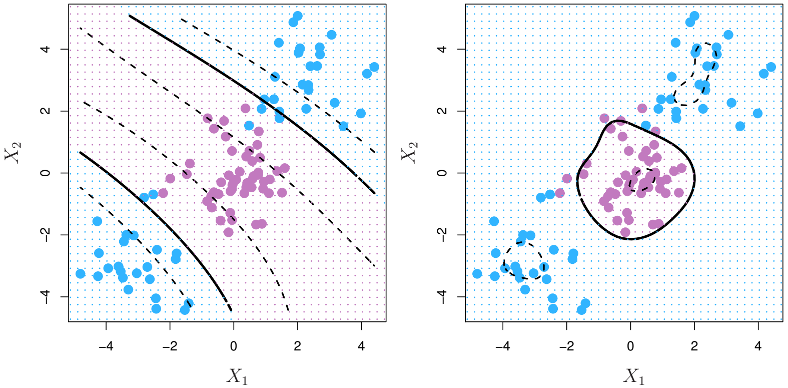
$$k(x_1, x_2) = \left(1 + x_1^\top x_2\right)$$

- Radial kernel

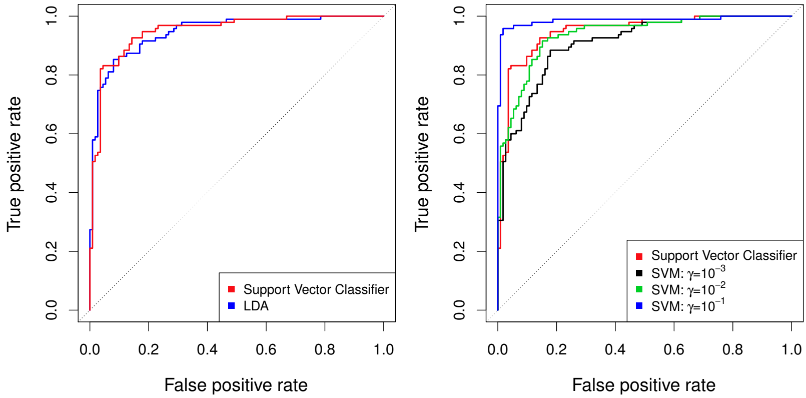
$$k(x_1, x_2) = \exp\left(-\gamma \|x_1 - x_2\|_2^2\right)$$

- Many many more

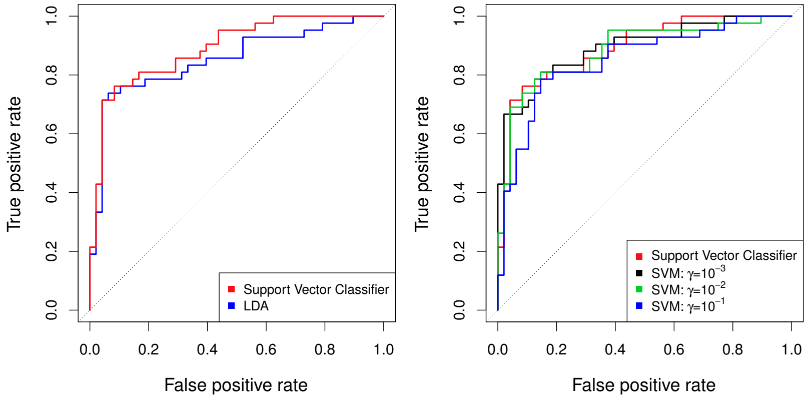
Polynomial and Radial Kernels



SVM vs LDA: Train



SVM vs LDA: Test



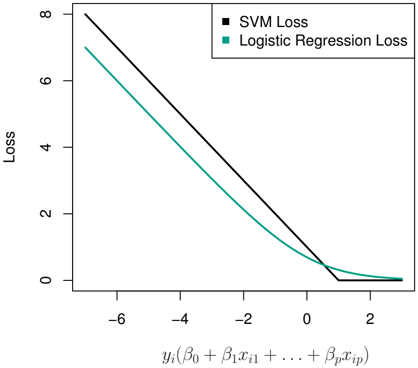
Multiple Classes

- One-vs-one

- One-vs-all

SVM vs Logistic Regression

- **Logistic regression**: Minimize negative log likelihood
- **SVM**: Minimize hinge loss



Bottom line: use SVM when classes are better separated or there a good *kernel*