Linear Algebra

Introduction

Marek Petrik

3/23/2017

Many slides adapted from Linear Algebra Lectures by Martin Scharlemann

Midterm Results

- Highest score on the non-R part: 67 / 77
- Score scaling: Additive in order for at least of 20% class gets over 95%
- Independent scaling for graduate and undergraduate students
- **Not** like grading on a curve5
- Questions that were most often wrong
- Some of the same questions will be **on the final** (practice tests)

Methods Covered Until Now

- Supervised learning
  1. Linear regression
  2. Logistic regression
  3. LDA, QDA
  4. Naive Bayes
  5. Lasso
  6. Ridge regression
  7. Subset selection
- Unsupervised learning
  1. PCA
  2. K-means clustering
  3. Hierarchical clustering
  4. Expectation maximization

Important Concepts We Have Covered

- Training and test sets
- Cross-validation and leave-one-out
- Maximum likelihood
- Maximum a posteriori

Remainder of the Course

- In ISL:
  1. Support vector machines
  2. Boosting and bagging
  3. Advanced nonlinear features
- Not in ISL:
  1. Recommender systems
  2. Methods for time series analysis
  3. Reinforcement learning
  4. Deep learning
  5. Graphical models

Today: Linear Algebra

- Crucial in many machine learning algorithms
- Which ones?
  1. Linear regression
  2. Logistic regression
  3. LDA, QDA
  4. Naive Bayes
  5. Lasso
  6. Ridge regression
  7. Subset selection
  8. PCA
  9. K-means clustering
  10. Hierarchical clustering
  11. Expectation maximization

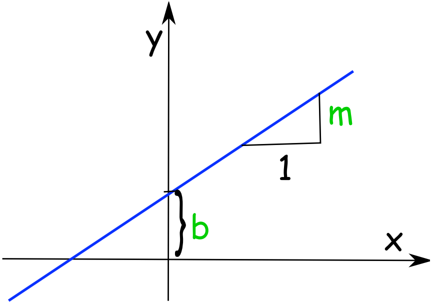
Suggested Linear Algebra Books

- Strang, G. (2016). Introduction to linear algebra (5th ed.)
  http://math.mit.edu/~gs/linearalgebra/
  Watch online lectures:
  https://ocw.mit.edu/courses/mathematics/18-06-linear-algebra-spring-2010/video-lectures/

- Hefferon, J. (2017). Linear algebra (3rd ed.).
  Free PDF:
  http://joshua.smcvt.edu/linearalgebra/

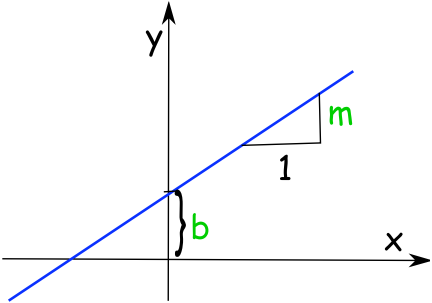
Linear Equation

Equation of a line:



Linear Equation

Equation of a line: $y = mx + b$



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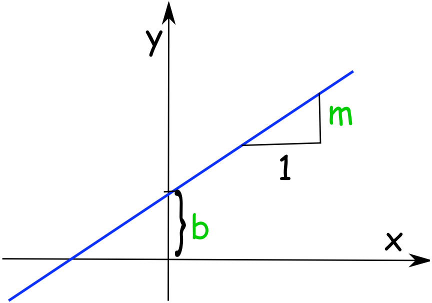

$$y = mx + b$$
$$\Updownarrow$$
$$y - mx = b$$

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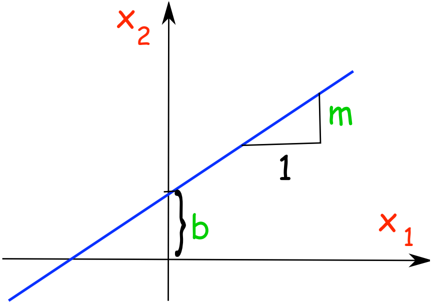

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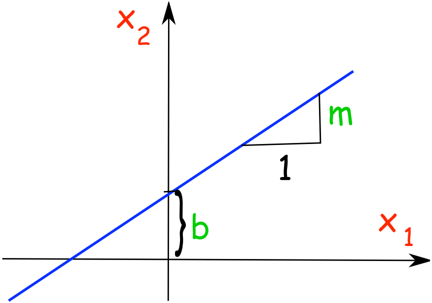

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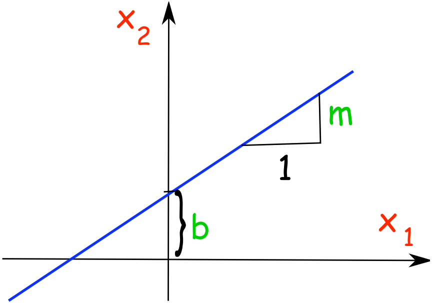

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$$x_2 - mx_1 = b$$
$$\Updownarrow$$
$$-mx_1 + x_2 = b$$
$$\Downarrow (\Uparrow a_2 \neq 0)$$
$$a_1x_1 + a_2x_2 = b'$$

Linear Equation

Linear equation in variables $x_1, ..., x_n$:

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where $a_1, ..., a_n$ and maybe b are all known in advance.

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Example 1: For the equation $a_1x_1 + a_2x_2 = b$ of a line, given above:

The pair s_1, s_2 is a solution $\iff$ the point (s_1, s_2) is on the line.

Example 2

Converting grades to the standard scale: $F = 0 \leq grade \leq 4 = A$, let

- x_1 be your first midterm grade
- x_2 be your second midterm grade
- x_3 be your grade on the final
- x_4 be your homework & quiz grade
- x_5 be your i-clicker grade

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Then

before then.

Grades:	Midterms @ 20%	40%
	Final	40%
	Homework and Quizzes	15%
	Class participation (via iClicker use)	5%

See "Discussion Sections" above for major penalty clause.
If this sounds too straightforward, consult the course Randomly Asked Questions page.

translates to

$$0.2x_1 + 0.2x_2 + 0.4x_3 + 0.15x_4 + 0.05x_5 = your\ course\ grade$$

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that is simultaneously a solution to all m equations.
That is, all m equations are true when $x_1 = s_1, x_2 = s_2, ..., x_n = s_n$.

Example

Conceptual example:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

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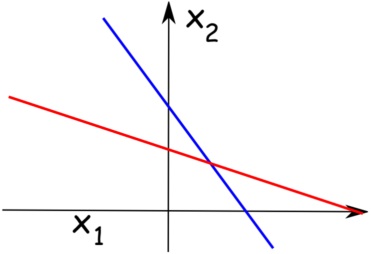

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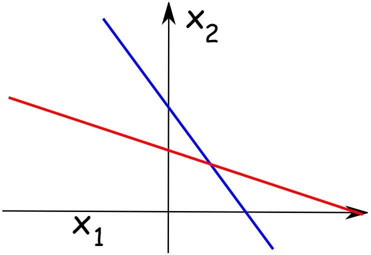
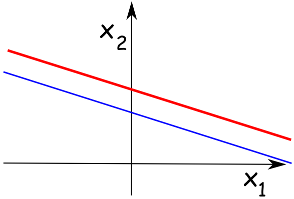


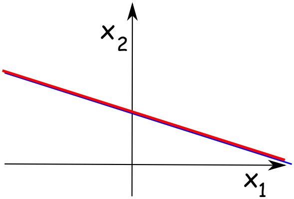
Example:

$$1x_1 + 2x_2 = 3$$
$$2x_1 + 1x_2 = 3$$
$$\Leftrightarrow (x_1, x_2) = (1, 1)$$

Line Configurations

Other possibilities:





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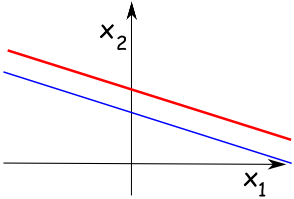
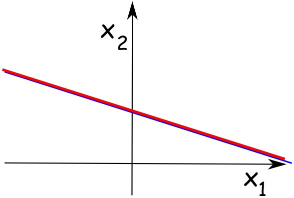


$$1x_1 + 2x_2 = 3$$

$$1x_1 + 2x_2 = 4$$

inconsistent



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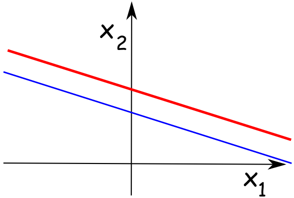

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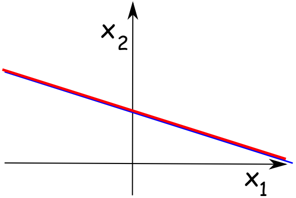

$$1x_1 + 2x_2 = 3$$
$$2x_1 + 4x_2 = 6$$

redundant

3 Possible Line Configurations

Upshot: There are exactly three possibilities

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- Write down the solution if it's unique.
- if there are infinitely many solutions, figure out a way to describe them all.

Matrix

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So extract just those numbers into a matrix

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$m \times n$ coefficient matrix

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$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$$

$m \times (n+1)$ augmented matrix

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Grand strategy: Do this until the equations are easy to solve.

Example

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$$2x_1 + 1x_2 = 3$$

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Step one:

Use first x_1 term [upper left matrix entry] to eliminate all other x_1 terms [change rest of first column to 0]:

$$\begin{aligned}
L1: \quad x_1 - 3x_2 - 2x_3 &= 6 \\
L2: \quad 2x_1 - 4x_2 - 3x_3 &= 8 \\
L3: \quad -3x_1 + 6x_2 + 8x_3 &= -5
\end{aligned}$$

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Subtract $2\times$ first from second
Add $3\times$ first to third:

$$\begin{aligned} L1: &\quad x_1 - 3x_2 - 2x_3 = 6 \\ L2 - 2L1: &\quad 2x_2 + x_3 = -4 \\ 3L1 + L3: &\quad -3x_2 + 2x_3 = 13 \end{aligned}$$

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Move on to Step two! - the second column

Step two:

$$\begin{aligned} L1: \quad x_1 - 3x_2 - 2x_3 &= \quad 6 \\ L2: \quad 2x_2 + x_3 &= -4 \\ L3: \quad -3x_2 + 2x_3 &= \quad 13 \end{aligned}$$

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Add $\frac{3}{2}\times$ L2 to L1 and L3:

$$\begin{aligned} L1 + \frac{3}{2}L2: \quad x_1 \qquad - \frac{1}{2}x_3 &= \quad 0 \\ L2: \quad 2x_2 + x_3 &= \quad -4 \\ L3 + \frac{3}{2}L2: \quad \frac{7}{2}x_3 &= \quad 7 \end{aligned}$$

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$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & \frac{7}{2} & 7 \end{array}\right]$$

Multiply L3 by $\frac{2}{7}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Move on to Step three! - the third column

Step three:

$$\begin{aligned} L1: &\quad x_1 \qquad - \frac{1}{2}x_3 = & 0 \\ L2: &\qquad\quad 2x_2 + x_3 = & -4 \\ L3: &\qquad\qquad\quad x_3 = & 2 \end{aligned}$$

Subtract L3 from L2;
add $\frac{1}{2}$ L3 to L1:

$$\begin{aligned} L1 + \frac{1}{2}L3: &\quad x_1 \qquad = & 1 \\ L2 - \frac{1}{2}L3: &\quad 2x_2 \qquad = & -6 \\ L3: &\qquad\quad x_3 = & 2 \end{aligned}$$

Step three:

$$\begin{aligned} L1: &\quad x_1 \qquad - \frac{1}{2}x_3 = 0 \\ L2: &\qquad 2x_2 + x_3 = -4 \\ L3: &\qquad\qquad x_3 = 2 \end{aligned}$$

$$\left[\begin{array}{ccc:c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 2 & 1 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

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Multiply L2 by $\frac{1}{2}$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

We're done: $(x_1, x_2, x_3) = (1, -3, 2)$

What can go wrong?

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$$x_1 + 2x_2 = 3$$
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Subtract 2L1 from L2:

$$x_1 + 2x_2 = 3$$
$$0 = 0$$

redundant
SOLUTIONS INFINITE

Configurations

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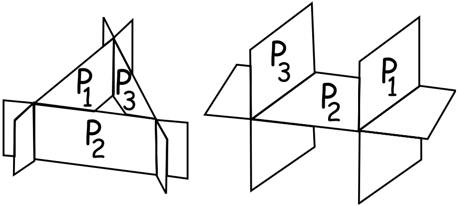

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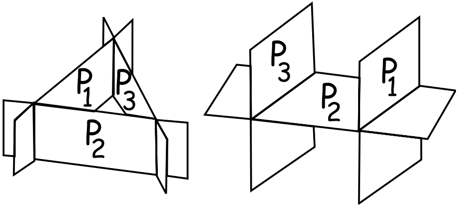

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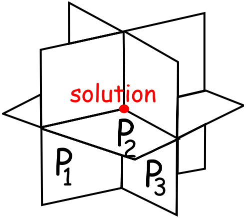
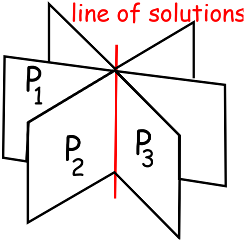


single solution



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Question?

George tells you the system of equations

$$\begin{aligned} 5x_1 + 2x_2 - 3x_3 &= 4 \\ 12x_1 - 7x_2 + 2x_3 &= 8 \\ -3x_1 + 4x_2 + 5x_3 &= 10 \end{aligned}$$

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B) George is probably wrong.
C) George is definitely right.
D) George is definitely wrong.
E) My brain is full.

Vector

An m-vector [column vector, vector in $\mathbb{R}^m$] is an $m \times 1$ matrix:

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_m \end{bmatrix}$$

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Do not multiply two vectors together like this.

Vector Operations

Examples:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix};$$

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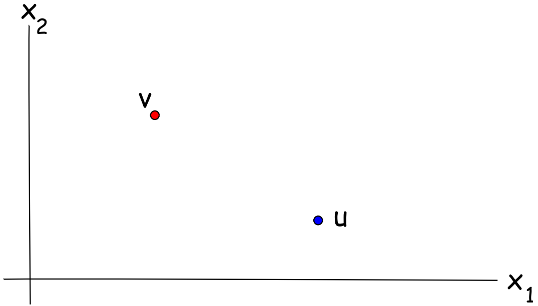
A) $\begin{bmatrix} 5 \\ -10 \\ -19 \end{bmatrix}$ B) $\begin{bmatrix} 5 \\ -9 \\ 10 \end{bmatrix}$ C) $\begin{bmatrix} 5 \\ 10 \\ 9 \end{bmatrix}$ D) $\begin{bmatrix} 5 \\ -19 \\ -10 \end{bmatrix}$ E) $\begin{bmatrix} 5 \\ 10 \\ 19 \end{bmatrix}$

Picturing Vectors

2- and 3-vectors:

- think of vector as arrow from 0
- multiply number with vector via scaling.
- add vectors head-to-tail;



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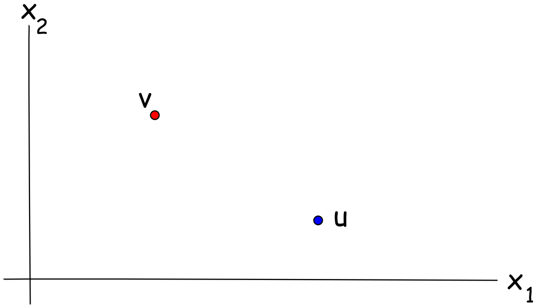

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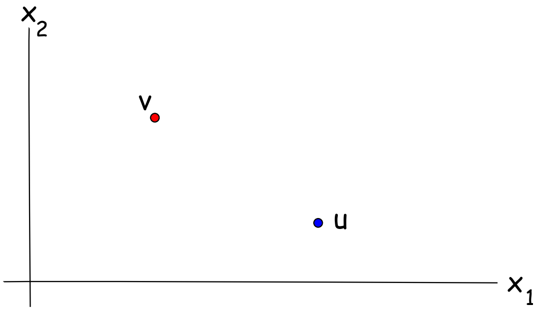

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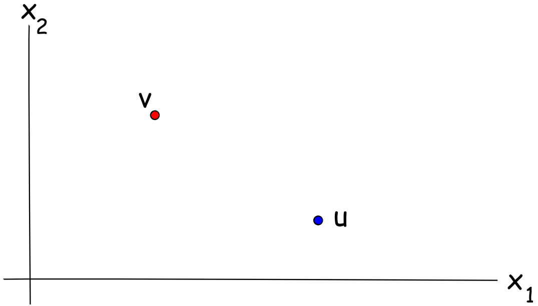

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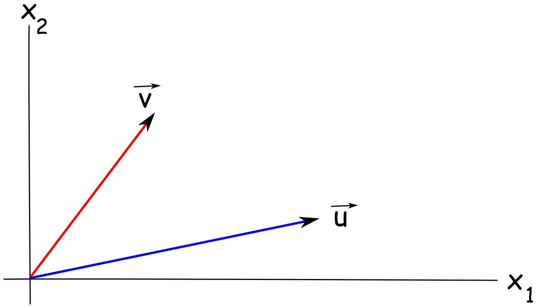

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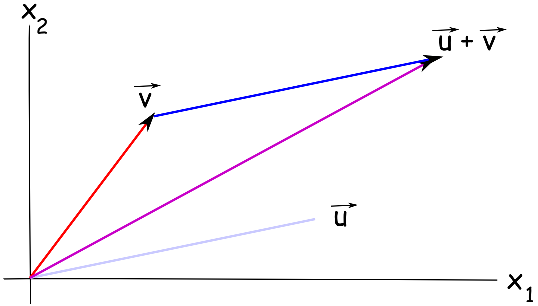

Picturing Vectors

2- and 3-vectors:

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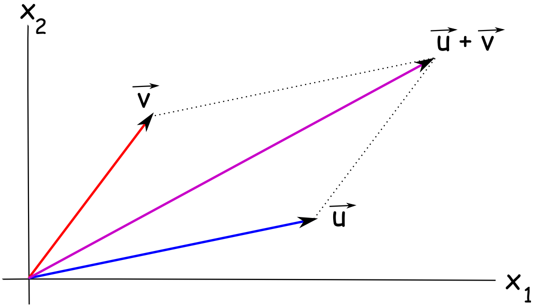

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Gives more flexible way to describe a line.

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For a line through a point p, in direction $\vec{d}$, use

$$\vec{x} = \vec{p} + t \cdot \vec{d}, \quad t \in \mathbb{R}$$

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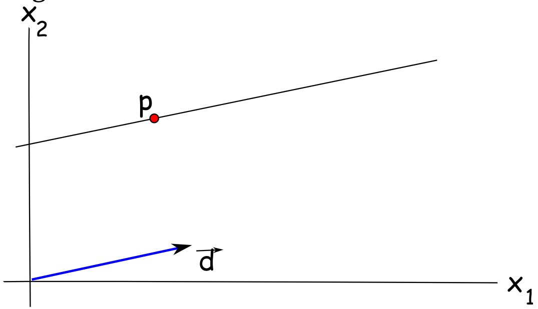

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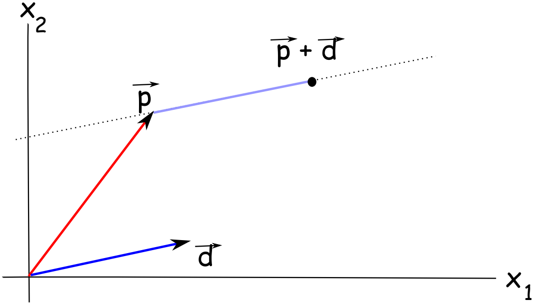

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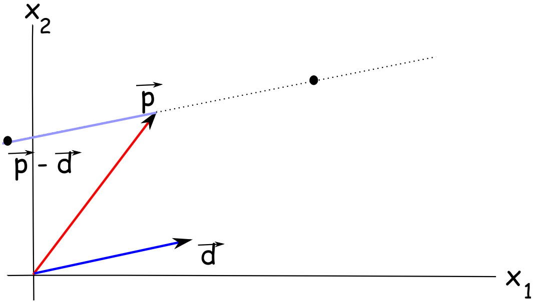

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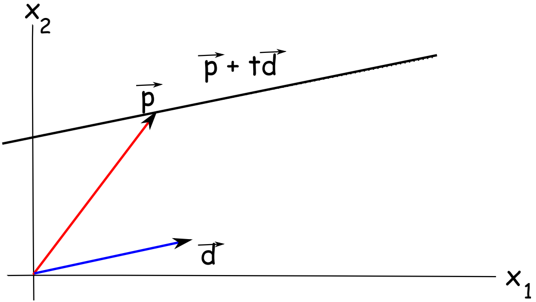

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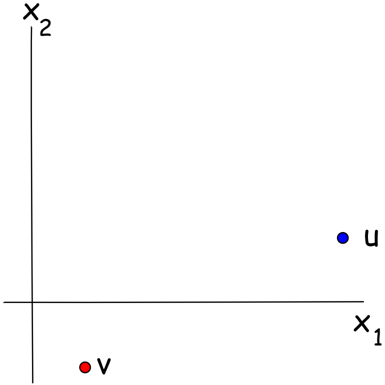

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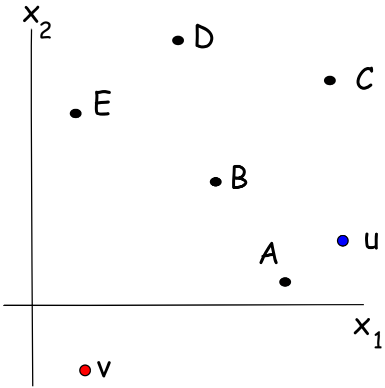

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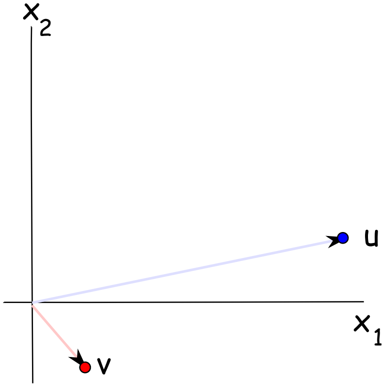

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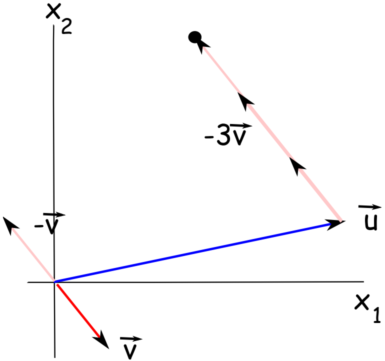

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Linear Combination

Definition

Suppose $\{t_1, t_2 \ldots t_k\}$ are all real numbers.
The vector

$$\vec{y} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$$

is called a linear combination of the vectors $\{\vec{v}_1, \vec{v}_2 \ldots \vec{v}_k\}$.

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Sample problem:
Given vectors $\{\vec{a}_1, \vec{a}_2 \ldots \vec{a}_n, \vec{b}\}$ in $\mathbb{R}^m$, find real numbers $\{t_1, t_2 \ldots t_n\}$ so that

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Off hand, could have any number of $\{t_1, t_2 \dots t_n\}$ solutions.

Linear Combination

How to think about solving for $\{t_1, t_2 \dots t_n\}$ in the equation

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Let

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}; \qquad \vec{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{mj} \end{bmatrix} \; for\; 1 \leq j \leq n$$

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Then for any $1 \le i \le m$, the i^{th} row of the equation becomes:

$$t_1a_{i1} + t_2a_{i2} + \cdots + t_na_{in} = b_i \ or$$

$$a_{i1}t_1 + a_{i2}t_2 + \cdots + a_{in}t_n = b_i$$

Linear Combination

In other words, solving

$$t_1\vec{a}_1 + \cdots + t_n\vec{a}_n = \vec{b}\,:$$

is the same as solving this system of m linear equations:

$$\begin{aligned} a_{11}t_1 + \ldots + a_{1n}t_n &= b_1 \\ a_{21}t_1 + \ldots + a_{2n}t_n &= b_2 \\ &\vdots \\ a_{m1}t_1 + \ldots + a_{mn}t_n &= b_m \end{aligned}$$

We just learned how to do this!

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$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right]$$

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Since each $\vec{a}_j$ is a column of i numbers, can just write

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n & \vec{b} \end{bmatrix}$$

Example

Suppose

$$\vec{a}_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 8 \end{bmatrix} \quad \vec{a}_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 8 \end{bmatrix} \quad \vec{a}_3 = \begin{bmatrix} 6 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \vec{a}_4 = \begin{bmatrix} 0 \\ 6 \\ 10 \\ 26 \end{bmatrix}$$

and want to find c_1, c_2, c_3, c_4 so that

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4 = \begin{bmatrix} 12 \\ 4 \\ 13 \\ 23 \end{bmatrix}$$

Example

This translates to the system of linear equations whose augmented matrix is

$$\left[\begin{array}{cccc:c} 0 & 0 & 6 & 0 & 12 \\ 2 & 2 & -1 & 6 & 4 \\ 4 & 4 & 1 & 10 & 13 \\ 8 & 8 & -1 & 26 & 23 \end{array}\right]$$

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which reduces to:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

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and so has general solution

$$c_2 = anything,\ c_1 = \frac{3}{2} - c_2, c_3 = 2, c_4 = \frac{1}{2}$$

Span

Definition

Given a collection $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ of vectors in $\mathbb{R}^m$, the set of all linear combinations of these vectors, that is all vectors that can be written as

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

for some $c_1, \ldots, c_k \in \mathbb{R}$ is denoted

$$\text{Span}\,\{\vec{v}_1, \ldots, \vec{v}_k\}$$

and is called the span of $\{\vec{v}_1, \ldots, \vec{v}_k\}$.

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Easy example: If $k = 1$ so there is only one vector $\vec{v}$, then $\mathrm{Span}\{\vec{v}\}$ is just all vectors that are multiples of $\vec{v}$. That is,
$\mathrm{Span}\{\vec{v}\} = \{c\vec{v} \mid c \in \mathbb{R}\}$

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Picturing the span when $m = 2, 3$:

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When there is only one vector $\vec{v}$ then $\mathrm{Span}\{\vec{v}\} = \{c\vec{v} \mid c \in \mathbb{R}\}$ is just the line that contains both $\vec{0}$ (take $c = 0$) and $\vec{v}$ (take $c = 1$).

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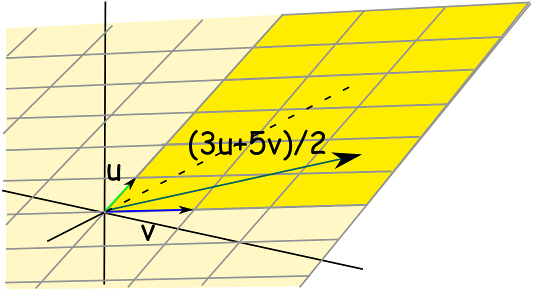
With two vectors $\vec{u}$ and $\vec{v}$, $\text{Span}\{\vec{u}, \vec{v}\} = \{c_1\vec{u} + c_2\vec{v}\}$ pictured via the parallelogram rule ($\text{Span} =$ entire plane; $c_i \geq 0$ highlighted):



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So we can think of the set of all solutions as

$$\begin{bmatrix} \frac{3}{2} \\ 0 \\ 2 \\ \frac{1}{2} \end{bmatrix} + \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

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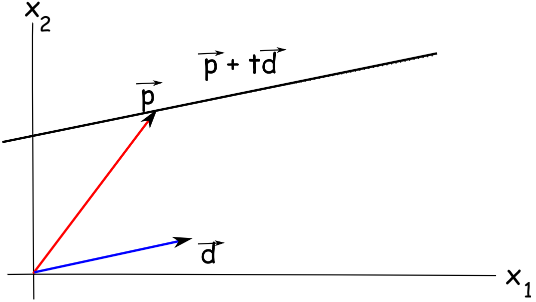
So we can picture the solution as a line in the direction of the second vector, going through the point given by the first vector (but in $\mathbb{R}^4$!)



Matrix Multiplication

Definition

The linear combination

$$x_1\vec{a}_1 + \cdots + x_k\vec{a}_k$$

is abbreviated

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \ldots & \vec{a}_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix}$$

.

Here $\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \ldots & \vec{a}_k \end{bmatrix}$ is the matrix A with i^{th} column $\vec{a}_i$.

Matrix Multiplication

Simplest example: each $\vec{a_i} \in \mathbb{R}^1$, i. e. each column in the matrix is just a number:

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_k \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots a_k b_k.$$

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Question

$$\begin{bmatrix} 1 & -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 2 \\ 1 \end{bmatrix} =$$

A) -3

B) -6

C) -9

D) 6

E) 9

Matrix-vector Multiplication

More complicated example: For $\vec{a_i} \in \mathbb{R}^2, i = 1, 2, 3$:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & -2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

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Think of doing the simple case on each row of the matrix A:

Matrix-vector Multiplication

Apply simple case to first row:

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Further abbreviation:

Use vector notation:

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$$x_1\vec{a}_1 + \cdots + x_k\vec{a}_k = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \ldots & \vec{a}_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_k \end{bmatrix} = A\vec{x}$$

Matrix-vector Multiplication

Summary: For A an $m \times n$ matrix, and a vector $\vec{x} \in \mathbb{R}^n$, multiplication $A\vec{x}$ is defined and gives a vector in $\mathbb{R}^m$.

Multiplication has two important properties:

- For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$

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For example:

$$A(\vec{u} + \vec{v}) = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \dots & \vec{a}_n \end{bmatrix} \left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \right) =$$

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$$(u_1+v_1)\vec{a}_1+\cdots+(u_n+v_n)\vec{a}_n = (u_1\vec{a}_1+\cdots+u_n\vec{a}_n)+(v_1\vec{a}_1+\cdots+v_n\vec{a}_n) =$$

$$A\vec{u} + A\vec{v}$$

Question

$$\begin{bmatrix} 1 & -2 & 3 & -4 \\ -2 & 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 2 \\ 1 \end{bmatrix} =$$

A) $\begin{bmatrix} 7 \\ -9 \end{bmatrix}$ B) $\begin{bmatrix} -9 \\ -4 \end{bmatrix}$ C) $\begin{bmatrix} -9 \\ 1 \end{bmatrix}$ D) $\begin{bmatrix} -4 \\ -9 \end{bmatrix}$ E) π^e

Matrix Equation

Sample problem from before:
Given vectors $\{\vec{a}_1, \vec{a}_2 \ldots \vec{a}_n\}$ and $\vec{b}$ in $\mathbb{R}^m$, find real numbers $\{x_1, x_2 \ldots x_n\}$ so that

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where $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \dots & \vec{a}_n \end{bmatrix}$. Note: the Matrix-vector multiplication $A\vec{x}$ makes sense only if the number of columns in A matches the number of entries in x.

Background

Background thoughts:

If two non-trivial vectors $\vec{x}_1, \vec{x}_2$ both lie on the same line, then $Span\{\vec{x}_1, \vec{x}_2\}$ is just that line.

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So the span of two vectors may be a plane, or it could be something simpler: either a line, or even just $\vec{0}$ in the case that $\vec{x}_1 = \vec{0} = \vec{x}_2$.

Background

Similarly, if *three* non-trivial vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ all lie on the same line, then $Span\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is just that line.

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If they don't all lie in the same plane, then $Span\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ looks like space.

Linear Independence

How do we put these ideas into math lingo, so we can be precise?

Definition

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^m$ is linearly independent if and only if the only solution to the equation

$$x_1\vec{v}_1 + \cdots + x_k\vec{v}_k = \vec{0}$$

is the solution $x_i = 0$ for $1 \leq i \leq k$.
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How do we put these ideas into math lingo, so we can be precise?

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A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^m$ is linearly independent if and only if the only solution to the equation

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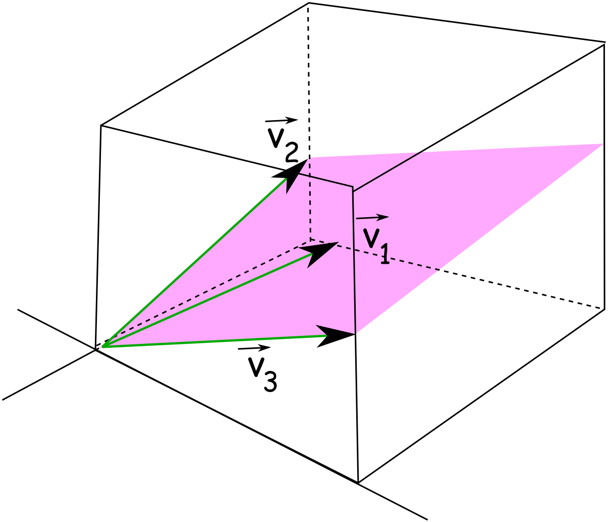
In pictures:



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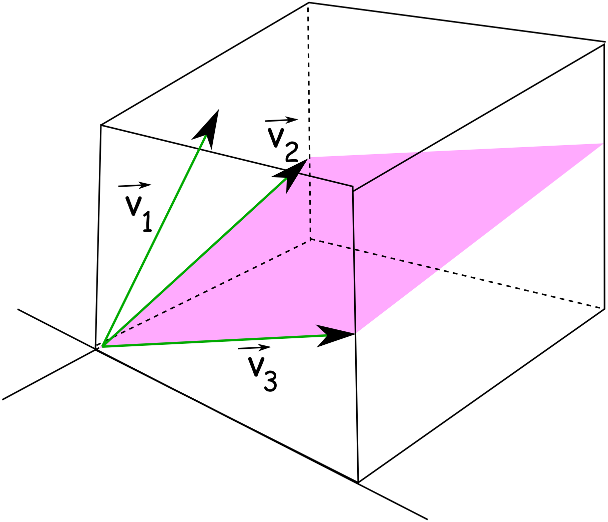

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Equivalently: if $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly independent then so is every subset of vectors from $\{\vec{v}_1, \ldots, \vec{v}_k\}$.

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Construct associated matrix of column vectors:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

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Here $6 > 5$ and x_3 is the free variable.

Question

Question: Is this set of vectors linearly dependent, or linearly independent?

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right\}$$

A) Dependent since they are 2 vectors in $\mathbb{R}^3$ and $2 < 3$.

B) Independent since they are 2 vectors in $\mathbb{R}^3$ and $2 < 3$.

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Answer: It's a pair of vectors and neither is a multiple of the other. Hence linearly independent.

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E) I can't tell.

Answer: For this triple of vectors

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} \right\}$$

we want to determine whether the homogeneous system of linear equations:

$$\begin{bmatrix} 1 & 1 & 4 \\ 1 & 3 & 9 \\ 0 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

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Since there is a free variable (namely x_3) there are non-trivial solutions, so linearly dependent.

Connection with Span

Theorem

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For example, suppose $\vec{v}_1 \in \mathrm{Span}\{\vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$.

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Since at least one of the coefficients (namely -1) is not zero, this shows the set of vectors is linearly dependent.

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Hence $\vec{v}_1 \in \mathrm{Span}\{\vec{v}_2, \vec{v}_3, \ldots, \vec{v}_k\}$.

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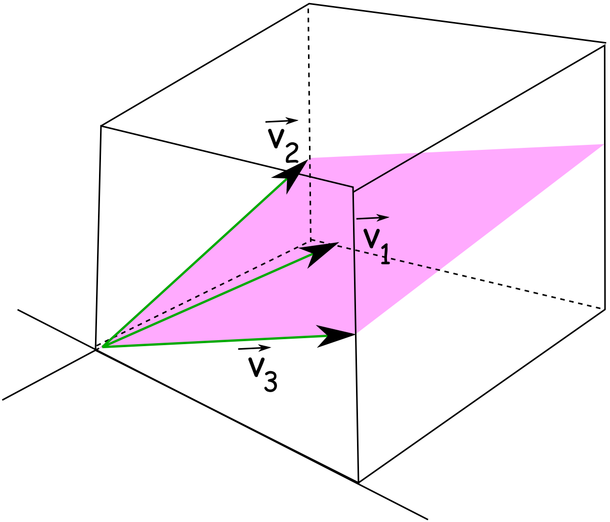
In pictures:



$\vec{v}_1 \in \mathrm{Span}\{\vec{v}_2, \vec{v}_3\}$ and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly dependent.

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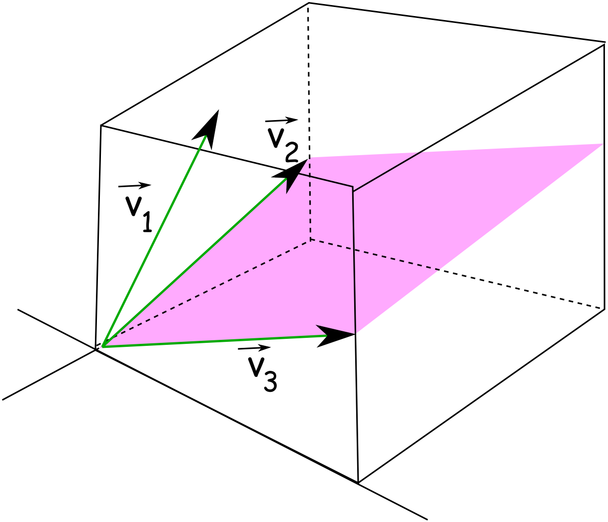

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