Blood Management Using Approximate Linear Programming

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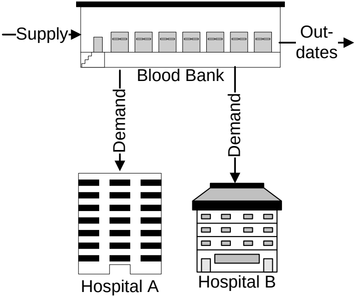
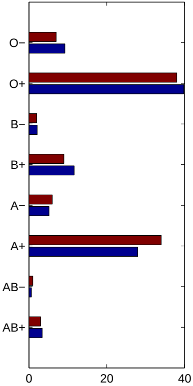
Blood Inventory Management Problem

- Regional blood banks:
  - Aggregate supplied blood
  - Supply demand requested by the hospitals
- Objectives:
  - Minimize **shortage** – demand that is not satisfied
  - Maximize **utilization** – amount of blood used before it perishes
  - Minimize **cost** – the financial cost of keeping the blood



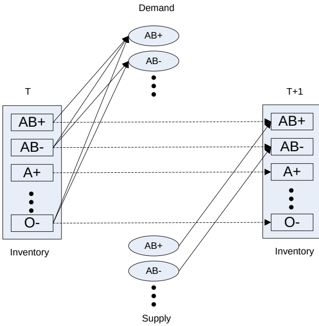
Considerations

- Demand and supply of blood are **stochastic**
- Blood is perishable
- Multiple blood types are compatible
- Blood type distribution: Supply $\neq$ Demand
- Manage how much of which blood is:
    1. Used to satisfy the demand
    2. Retained in inventory
- Challenges not addressed:
    - Large portion of blood that is reserved is not used
    - Usage depends on the hospital type



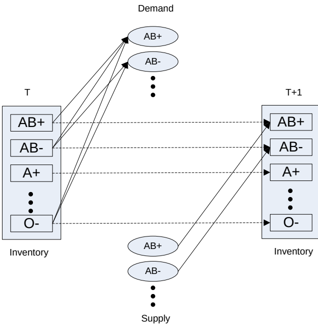
Formalization

- Multistage stochastic problem – stage $=$ week
- Decide whether to satisfy the demand or keep the inventory
- Formulate as a **Markov decision process**:
  - States: $\mathcal{S}$, Actions: $\mathcal{A}$
  - Transition function: $P(s_1, a, s_2)$ – probability of transition from s_1 to s_2 with action a
  - Contribution (reward) function: $r(s, a)$ for state s and action a:



Transition Function

- State:
  - **Inventory**: Available blood types and ages
  - **Demand**: Amount of blood required
- Action:
  - Blood amounts and types used satisfy the demands
- Transition function:
  - Stochastic demand
  - Stochastic supply added to inventory
  - Blood discarded after 5 weeks



Contribution (Reward) Function

- Determines tradeoffs in satisfying the demands:
- Contribution is **linear** per unit of blood demand

Type	Reward
Unsatisfied	0
Same type	50
Compatible type	45

- Contribution of using blood type i for blood type j: c_{ij}

Myopic Solution

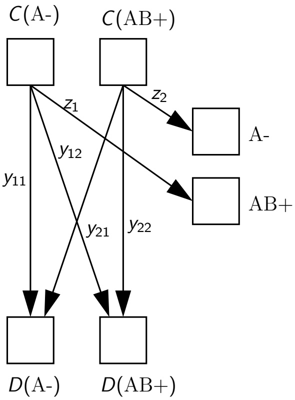
- Finding the best way of using a given inventory – single step
- Actions:
  - y_{ij} – Type i used to satisfy demand for type j
  - z_i – Type i that is retained in inventory
- Solved as a simple flow problem:

$$\max_{y,z} \quad \sum_{ij} c_{ij} y_{ij}$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{T}} y_{ij} + z_k \leq C(i) \quad \forall i \in \mathcal{T}$$

$$\sum_{i} y_{ij} \leq D(j) \quad \forall j \in \mathcal{T}$$

$$y_{ij}, z_i \geq 0 \quad \forall i, j \in \mathcal{T}$$



Myopic Solution – Performance

- Performance in infinite-horizon discount setting?
- **Myopic solution:** $\approx$ 273 000
  - Uses all the supply
  - Often significant shortages

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Myopic Solution – Performance

- Performance in infinite-horizon discount setting?
- **Myopic solution:** $\approx 273\ 000$
  - Uses all the supply
  - Often significant shortages
- **Optimal solution**: $\approx 275\ 000$
- Explanation:
  - Linear penalty for running out of blood
  - Small variance in supply and demand.
- Conclusion:
  - No sophisticated inventory management necessary to satisfy the supply
  - Model does not justify keeping blood inventory

Modified Contribution Function

- Include the emergency of the blood request
  - Critical
  - Urgent
  - Elective
- Contribution function is:

Type	Critical	Urgent	Elective
Unsatisfied	0	0	0
Same type	50	25	5
Compatible type	45	27.5	4.5

- Need to keep inventory:
  - **Myopic solution**: $\approx$ 70 000
  - **Optimal solution**: $\approx$ 94 000
- Myopic solution is suboptimal

Infinite Horizon Objective

- Optimize over infinite number of steps (weeks)
- Start with an initial state s_0
- The reward is discounted with $\gamma = 0.9$:

$$\mathbf{E}_{s_0}\left[\sum_{i=0}^{\infty} \gamma^i R_i\right] = \mathbf{E}\left[R_0 + 0.9 R_1 + 0.9^2 R_2 + 0.9^3 R_3 + \ldots\right]$$

- **Value function:** $v(s)$
  - Assigns value to each state s
  - Discounted return when starting in state s:

$$\mathbf{E}_{s}\left[\sum_{i=0}^{\infty} \gamma^i R_i\right] = \mathbf{E}\left[R_0 + 0.9 R_1 + 0.9^2 R_2 + 0.9^3 R_3 + \ldots\right]$$

Value Function as Linear Program

- Constraints:

$$v(s_1) \geq \gamma p\left(\cdot|s_1, a_1\right)^{\mathsf{T}} v + r_{a_1}$$
$$v(s_1) \geq \gamma p\left(\cdot|s_1, a_2\right)^{\mathsf{T}} v + r_{a_2}$$
$$\vdots$$

- That is:

$$v(s_1) \geq \max_{a \in \mathcal{A}} \mathbf{1}_{s_1}^{\mathsf{T}} \left(\gamma P_a v + r_a\right)$$

- For any feasible solution v we have $v \geq v^*$
- Minimal feasible solution is v^*
- Linear program:

$$\begin{aligned} \min_{v} \quad & \mathbf{1}^{\mathsf{T}} v \\ \text{s.t.} \quad & A v \geq b \end{aligned}$$

- **Problem:** Too large to solve optimally

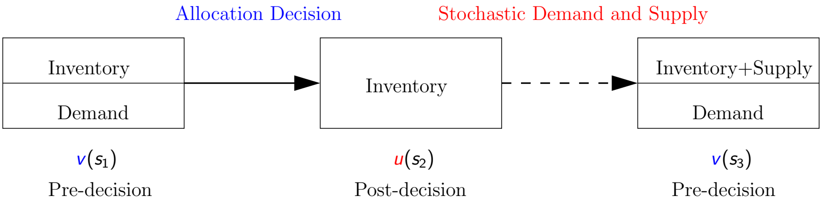
Approximate Linear Program

- Reduces number of variables in the LP
- Consider an **approximation basis**: M, as a matrix
- Value function from *colspan*(M): $v = Mx$
- Approximate linear program:

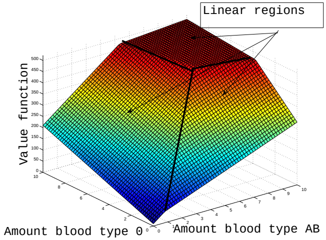
$$\begin{aligned} \min_{v} \quad & \mathbf{1}^{\mathsf{T}} M x \\ \text{s.t.} \quad & AMx \geq b \end{aligned}$$

- Many constraints – reduce by *sampling*
- Better theoretical properties than other approximate methods – finds the optimal solution if it is representable

Value Function in Blood Inventory Management



Value function:



- Two types of value function:
  1. u value of post-decision state
  2. v value of pre-decision state
- Greedy step:

$$\arg\max_{a \in \mathcal{A}} \mathbf{1}_s^{\mathsf{T}} \left(\gamma P_a u + r_a\right)$$

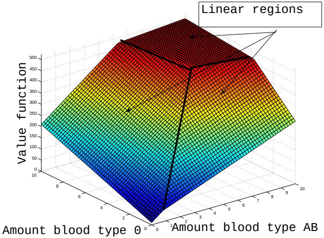
Approximation Basis in Blood Inventory Management

- Defines a set of values for each post-decision state – inventory.
- Structure:
  - Piece-wise linear
  - Fixed regions of linearity
- $M =$

	Feature A	Feature B
A=0, B=1	0	1
A=0, B=2	0	2
A=1, B=0	1	0
A=2, B=0	2	0
A=1, B=1	1	1

- Greedy step be formulated as a flow problem

Example value function:



Blood Inventory Management ALP

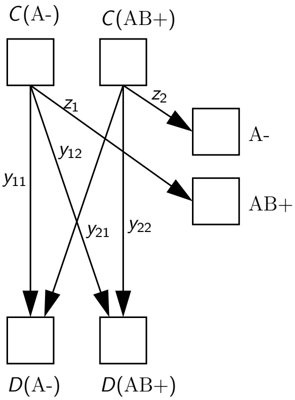
- ALP Constraints:

$$u(s_1) \geq \mathbf{1}_{s_1}^{\mathsf{T}} \left(\gamma P_{a_1} u + r_{a_1}\right)$$
$$u(s_1) \geq \mathbf{1}_{s_1}^{\mathsf{T}} \left(\gamma P_{a_2} u + r_{a_2}\right)$$
$$\vdots$$

- But $|\mathcal{A}| = \infty$; use:

$$u(s_1) \geq \max_{a \in \mathcal{A}} \mathbf{1}_{s_1}^{\mathsf{T}} \left(\gamma P_a u + r_a\right)$$

$$\begin{aligned} u(s_1) \geq \max_{y,u,z} \quad & c^{\mathsf{T}} y \\ \text{s.t.} \quad & A_1 y + A_2 z \leq b_1 \\ & By \leq b_2, \; y, z \geq \mathbf{0} \end{aligned}$$



- **Problem:** *Not* a linear program

Dual Formulation of Blood Inventory Management

- Dualize to get a linear program

$$\begin{aligned} u(s_1) \geq \min_{\lambda_1,\lambda_2} \quad & b_1^\mathsf{T}\lambda_1 + b_2^\mathsf{T}\lambda_2 \\ \text{s.t.} \quad & A_1^\mathsf{T}\lambda_1 + B^\mathsf{T}\lambda_2 \geq c \\ & A_2^\mathsf{T}\lambda \geq u \\ & \lambda_1, \lambda_2 \geq \mathbf{0} \end{aligned}$$

- Leads to:

$$\begin{aligned} \min_{u,\lambda_1,\lambda_2} \quad & u(s_1) + u(s_2) + \dots \\ \text{s.t.} \quad & u(s_1) \geq b_1^\mathsf{T}\lambda_1 + b_2^\mathsf{T}\lambda_2 \\ & A_1^\mathsf{T}\lambda_1 + B^\mathsf{T}\lambda_2 \geq c \\ & A_2^\mathsf{T}\lambda \geq u \\ & \lambda_1, \lambda_2 \geq \mathbf{0} \end{aligned}$$

Performance and Approximation Error

- **ALP Solution quality**: 18 000

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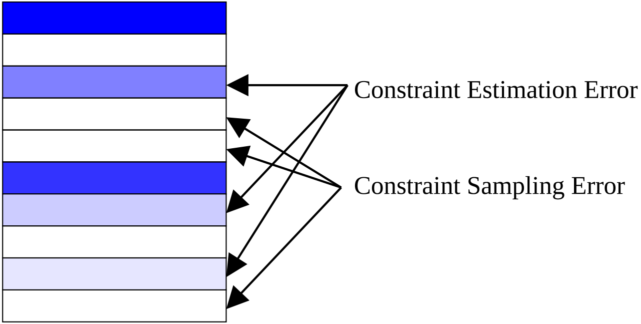
- **Myopic solution:** 70 000

Performance and Approximation Error

- **ALP Solution quality**: 18 000
- **Myopic solution:** 70 000
- Approximation is too loose
- Approximation errors:
    1. *Representational* – Limited approximation features (basis) M
    2. *Transitional* – Limitation of ALP formulation
    3. *Sampling* – Limited number of sampled constraints

Sampling Error

- Sources:
  1. Constraint sampling
  2. Constraint estimation
- Constraint matrix A:



Constraint Sampling

- **Full ALP:**

$$\min_x \quad \mathbf{1}^\mathsf{T} Mx$$
$$\text{s.t.} \quad \mathbf{1}_s^\mathsf{T} Mx \geq \mathbf{1}_s^\mathsf{T}(\gamma P_a Mx + r_a) \quad \forall s \in \mathcal{S}$$

- Constraint for each state in $\mathcal{S}$
- Consider a subset $\tilde{\mathcal{S}} \subset \mathcal{S}$
- **Reduced ALP:**

$$\min_x \quad \mathbf{1}^\mathsf{T} Mx$$
$$\text{s.t.} \quad \mathbf{1}_s^\mathsf{T} Mx \geq \mathbf{1}_s^\mathsf{T}(\gamma P_a Mx + r_a) \quad \forall s \in \tilde{\mathcal{S}}$$

Constraint Estimation

- Constraints in ALP:

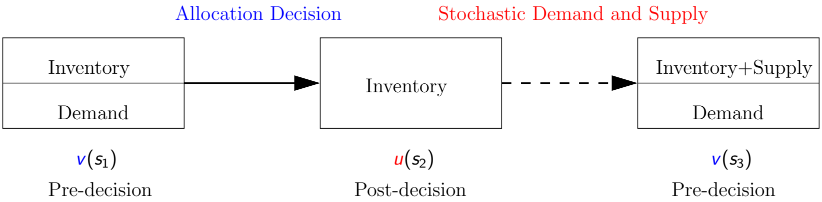
$$\mathbf{1}_s^\mathsf{T} Mx \geq \mathbf{1}_s^\mathsf{T} \gamma P_a Mx + r_a) \quad \forall s \in \mathcal{S}$$

- Must be sampled when:
  - Unknown problem model
  - Possible transition to too many states
- Sample states from the transition probability $s \to s_1, s_2, \ldots, s_n$
- Constraint:

$$v(s) \geq \gamma P_a v + r_a = \gamma \mathrm{E}_S \left[v(S) \right] + r_a$$
$$\approx \gamma \frac{1}{n} \sum_{j=1}^{n} v(s_j) + r_a$$

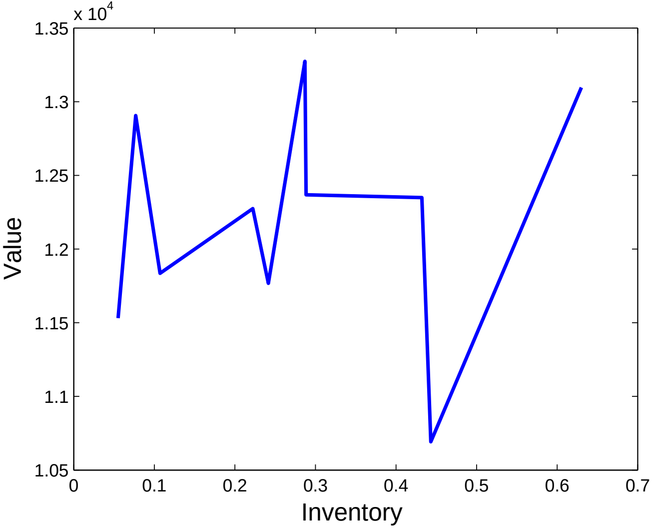
- Can show regularity for ALP – for sufficiently large n, the error is sufficiently small
- The number of samples depends on the number of features in the ALP

Constraint Sampling



- Constraint sampling = selecting the inventory
- Constraint estimation = selecting stochastic supply and demand
- **Problem:** The stochastic demand and supply effect is larger that the demand effect – the variance is too high

Constraint Sampling Error



Synchronized Sampling

- Exploit:
  - Inventory influence mostly independent of the demand and supply
- Use ω to denote the stochastic supply/demand
- $f(s, \omega) =$ the state that follows from s given action a and demand/supply ω

Synchronized Sampling

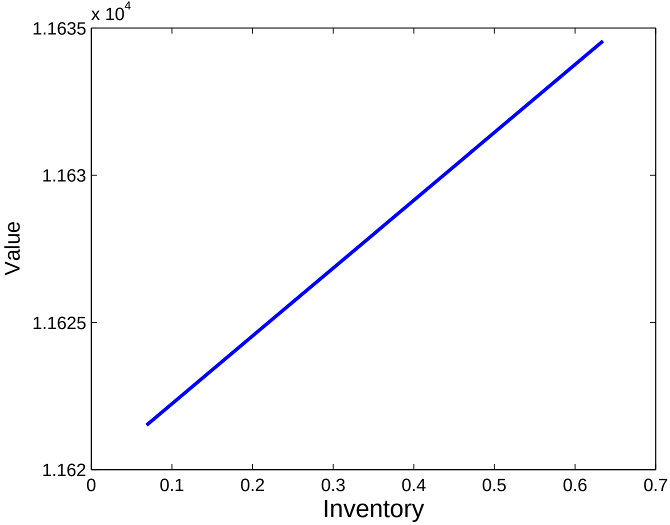
- Sampled supply/demand: $\omega_1^1, \omega_2^1, \ldots, \omega_1^2, \omega_2^2, \ldots$
- Standard constraint sampling

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ & & \vdots & \\ 0 & 0 & 0 & \ldots 1 \end{pmatrix} - \gamma \frac{1}{n} \begin{pmatrix} -- & \sum_{j=1}^{n} v(f(s_1, \omega_j^1)) & -- \\ -- & \sum_{j=1}^{n} v(f(s_2, \omega_j^2)) & -- \\ -- & \vdots & \end{pmatrix}$$

- Synchronized constraint sampling

$$A = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ & & \vdots & \\ 0 & 0 & 0 & \ldots 1 \end{pmatrix} - \gamma \frac{1}{n} \sum_{j=1}^{n} \begin{pmatrix} -- & v(s_1) - \gamma v(f(s_1, \omega_j)) & -- \\ -- & v(s_2) - \gamma v(f(s_2, \omega_j)) & -- \\ -- & \vdots & -- \end{pmatrix}$$

Synchronized Constraint Sampling Error

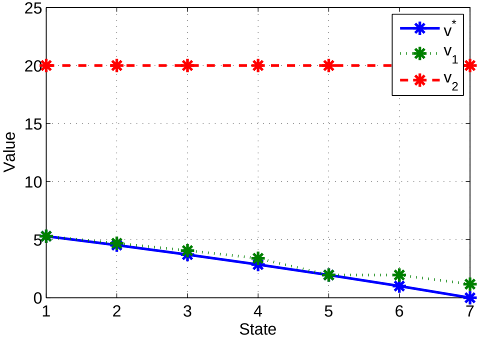


Transitional Error

- The standard bound:

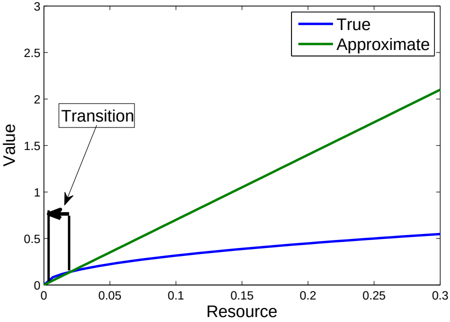
$$\|v^* - \tilde{v}\|_1 \leq \frac{2}{1-\gamma} \min_x \|v^* - Mx\|_\infty$$

- Approximate value function may still be useless:



Resource Management Transitional Error

- Typically concave value functions
- Approximation by a piece-wise linear function
- ALP can be seen as approximating the derivative of the value function
- Consider an MDP with value function:



Relaxed Approximate Linear Program

- **Allow limited constraint violation**
- Original linear program:

$$\begin{aligned} \min_{v} \quad & \mathbf{1}^{\mathsf{T}} v \\ \text{s.t.} \quad & Av \geq b \end{aligned}$$

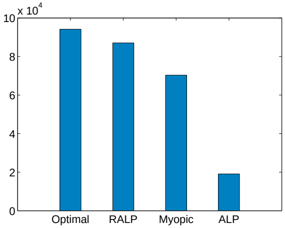
- Assume a weight distribution on the constraints: d
- Transformed into:

$$\min_{v} \quad \mathbf{1}^{\mathsf{T}} v + d^{\mathsf{T}} [r - Ax]_{+}$$

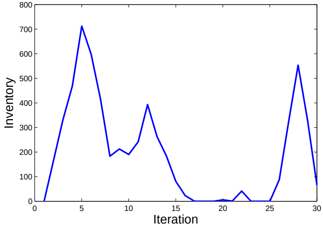
- Corresponds to upper bounds on dual variables
- If $d \geq \frac{1}{1-\gamma}\mathbf{1}$ the solution is identical to ALP
- Preserves good theoretical properties of ALP

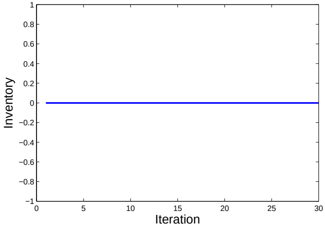
Empirical Performance

- Performance of ALP:



- Inventory level:





Conclusion

- Blood inventory management is an interesting and hard resource management problem
- Payoff for blood supply must be concave or the optimal solution is trivial – myopic
- Important aspects of ALP solution:
  - The sampling method for constraint selection
  - The sampling method for constraint estimation
  - Relaxation of "outlier" constraints
- ALP can work well
- **But** needs significant tweaking and adjustments